A framework for determining Earth's magnetic field

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Abstract

It is understood that the geomagnetic field, which is determined by a series of significant physical and chemical processes that occurred within the region of Earth's core, mantle, crust, ocean and space, is stable and persistent in large spatial scales in long-term but strongly dynamical in smaller scales, especially during the period of the magnetic storms. Limited by the geomagnetic observations and the complexity of Earth's dynamical systems, the total geomagnetic field inversion is often performed in the near-Earth region using the observational data in the magnetic quiet time with the simplified (quasi-)linear assumptions for the contribution of the source terms.

One of the ultimate goals of this development is to utilise the Macau science satellite (MSS-1) and other high-quality observational data, e.g., Swarm, to better understand the dynamics of Earth's geomagnetic systems, quantify the uncertainty of different geomagnetic sources and more accurately model the total geomagnetic field. In this presentation, I will discuss an effective approach to incorporate the conventional geomagnetic field model with more accurate physical constraints for simultaneously separating different source terms from the total field and probing the inner workings of Earth's magnetic systems, which are not directly observable.

1. The conventional description of Earth's magnetic field

IN conventional approach for modelling the geomagnetic field, the current-free approximation is often employed for describing the near-Earth electromagnetic environments. The geomagnetic field, $\mathbf{B}$, in this region may then be accurately represented by a scalar potential function, Ψ, i.e.,

$$\mathbf{B} = -\nabla\Psi. \tag{1}$$

The scalar potential, Ψ, is found to satisfy the Laplace equation, $\nabla^2\Psi = 0$, as the divergence of Eq. (1) vanishes, $\nabla\cdot\mathbf{B} = 0 = -\nabla^2\Psi$. In spherical polar coordinates, (r,θ,ψ), the general solution, Ψ, reads

$$\Psi = \sum_{l,m}\left(G_l^m(t)\frac{1}{r^{l+1}} + Q_l^m(t)r^l\right)Y_l^m(\theta,\psi). \tag{2}$$

The function, Y_l^m, is known as the spherical harmonics and the spectral coefficients, $\{G_l^m, Q_l^m\}$, are termed Gauss coefficients, which represent the internal and external contribution of the total field, respectively.

The Gauss coefficients associated with different physical processes in characteristic spatiotemporal scales can be further written as

$$G_l^m(t) = \underbrace{g_l^m(t)}_{\text{core field}} + \underbrace{d_{st}(t)h_l^m}_{\text{mantel induction}} + \underbrace{\left\{\begin{array}{l}\cos(\omega t)e_l^m\\ \sin(\omega t)e_l^m\end{array}\right.}_{\text{tides effect}} + \cdots, \tag{3}$$

where d_{st} index measures the variation of the geomagnetic field in the equatorial region and ω is the frequency of ocean tides.

We note that the spatiotemporal variation of the geomagnetic field is determined by a set of dynamical systems operating outside the current-free region; whilst the scalar potential representation only describes the geomagnetic field within the current-free region.

2. The downward and upward continuation of Earth's magnetic field beyond the current-free region

GIVEN a time series of Gauss coefficients, we may describe the geomagnetic field variation within the current-free region between the inner and outer boundaries. In a short time window, Earth's core field can be approximated by a steady field and the magnetic field generated by the rapid variation of the ring current and the induced mantel field is the main contribution to the temporal variation of the total field [1]. Specifically,

- by neglecting the electromagnetic effects of Earth's ionosphere, the scalar potential field at the outer boundary is found to match the geomagnetic field generated by the magnetohydrodynamical system of the magnetosphere at $r = r_1$ with the governing equation given by

$$\begin{aligned}\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} &= -\nabla p + \mathbf{J}\times\mathbf{B},\\ \partial_t\mathbf{B} &= \nabla\times(\mathbf{u}\times\mathbf{B}),\end{aligned} \tag{4}$$

  where $\mathbf{u}$ and $\mathbf{J}$ are the velocity field and electrical current of the magnetosphere, respectively;
- by assuming the finite conductivity of Earth's mantle, the scalar potential field may be projected downward to the surface of Earth at $r = r_0$. If the magnetic field of the magnetosphere varies rapidly, the induced magnetic field of Earth's mantel governed by

$$\partial_t\mathbf{B} = -\nabla\times(\eta\nabla\times\mathbf{B}) \tag{5}$$

is strong enough to be observed. The magnetic field in the magnetosphere, mantel and the current-free region are coupled. For example, at the surface of Earth, the poloidal scalar, S, of the mantel magnetic field is determined by the Gauss coefficient of the ring current, q_l^m, and reads

$$\frac{dS}{dr} + \frac{l+1}{r}S = -\frac{2l+1}{l+1}q_l^m r^{l-1}. \tag{6}$$

3. The partial differential equation constrained optimisation

WE optimise the Gauss coefficient to best fit the geomagnetic observations by minimising a non-negative objective functional, $\mathcal{J}$, given by

$$\mathcal{J} = \frac{1}{2}\sum_j\left[H(\mathbf{B}_j)-\mathbf{y}_j\right]^T\cdot e_{rr}^{-1}\cdot\left[H(\mathbf{B}_j)-\mathbf{y}_j\right] + \left\langle\mathbf{B}^\dagger, \partial_t\mathbf{B} + \nabla\times(\eta\nabla\times\mathbf{B})\right\rangle_{\text{mantel}} + \cdots, \tag{7}$$

where H and $\mathbf{y}$ are the observational operator and the magnetic observations, e_{rr}, is the error covariance and $\mathbf{B}^\dagger$ is the adjoint field of Earth's mantel, see e.g., [2].

A number of optimisation methods are available for minimising $\mathcal{J}$, e.g.,

- the gradient-based Newton's methods & conjugate-gradient method or
- the Bayesian statistical method.

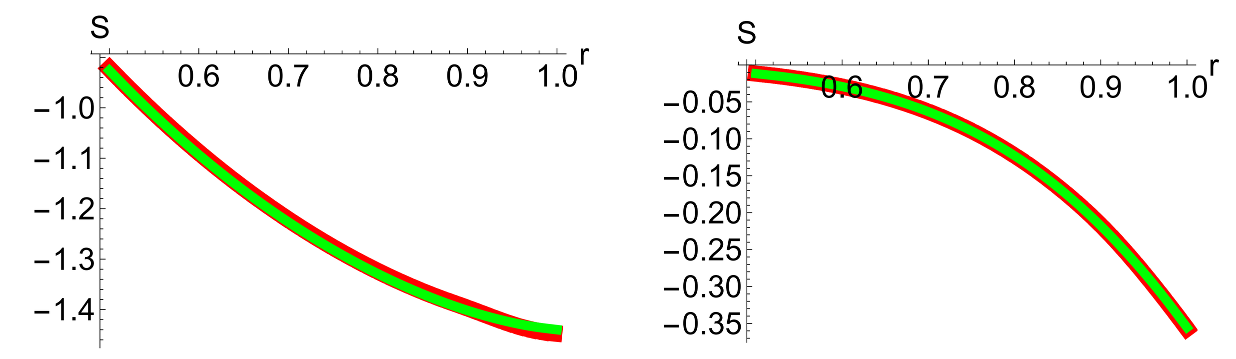
In this study, we compute the gradient of $\mathcal{J}$, i.e.,

$$\nabla\mathcal{J} = \left[\frac{\delta\mathcal{J}}{\delta g_l^m}, \frac{\delta\mathcal{J}}{\delta q_l^m}, -\mathbf{B}^\dagger(t=0), \cdots\right] \tag{8}$$

and apply the **quasi-Newton** (L-bfgs) method for minimising $\mathcal{J}$ to obtain the optimal Gauss coefficients and the evolution of mantel magnetic field within the observation time window. The complete methodology for solving the adjoint system may be found in an abundance of literature, e.g., see [3] for details. A numerical algorithm similar to [4] is created for solving the mantel induction problem by projecting the induction equation onto the spectral space of the spherical harmonics and discretising the radial equations via the finite element method.

4. The closed-loop test

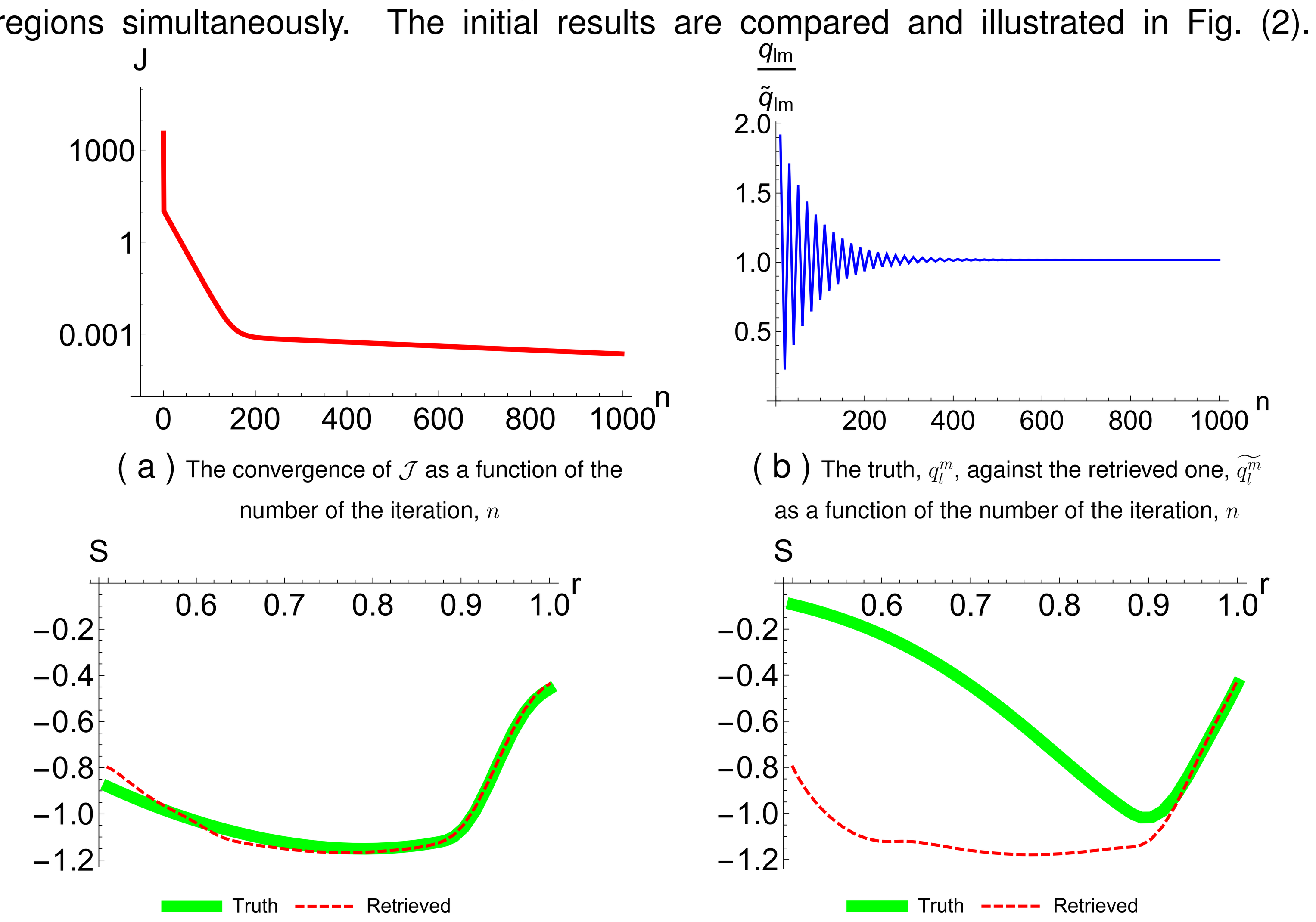
TO simplify the initial development, we assume a constant mantel conductivity to decouple the poloidal and toroidal components from the induction equation. Fig. (1) illustrates a benchmark case of the equilibrium state of the poloidal scalar functions obtained by using the Chebyshev-collocation and finite element method in the spectral space, (l,m), for $l = 1$ and 5 with a prescribed d_{st} index.



(a) for $l = 1$ case (b) for $l = 5$ case

Figure 1: *The poloidal radial scalars obtained via collocation method in red and finite element method in green.*

We use a prescribed core field, g_l^m, ring current field, q_l^m and the mantel induction equation to generate a set of synthetic data in space and time to mimic the satellite observations without considering the observational or model error and apply the optimisation framework in (8) to retrieve the geomagnetic field in the current-free and the mantel regions simultaneously. The initial results are compared and illustrated in Fig. (2).



(a) The convergence of $\mathcal{J}$ as a function of the number of the iteration, n

(b) The truth, q_l^m, against the retrieved one, $\widetilde{q_l^m}$ as a function of the number of the iteration, n

(c) The retrieved initial condition of the magnetic induction equation *vs.* the truth for $l = 1$.

(d) The retrieved initial condition of the magnetic induction equation *vs.* the truth for $l = 5$.

Figure 2: *The numerical performance of the optimisation framework.*

5. Remarks and conclusions

To best incorporate with the conventional geomagnetic field model,

- we choose to descretise the induction equation of Earth's mantel in the spectral space defined by the **spherical harmonics**.

To cope with the discontinuity and the heterogeneity of the mantel conductivity, η,

- a **finite element** approach is developed for solving the radial component of the diffusion equations.

A hybrid optimisation framework is developed for retrieving

- the 2D Gauss coefficients, e.g., g_l^m & q_l^m, and
- the 3D dynamical systems occurred outside the current-free region **simultaneously**.

We find that both 2D and 3D fields can be accurately reconstructed using satellite observations.

We are working on

- a systematic study for understanding the long-standing problems in geomagnetism ...
- the further developments of the framework and ....

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