The Spatial Race Between Labour and Capital: A Route to Turing Instability

Davide Fiaschi **Cristiano Ricci**

Department of Economics and Management
University of Pisa, Italy

February 28, 2026

Abstract

A central question in spatial and growth economics is whether the mobility of production factors across locations constitutes a force for convergence or divergence. The neoclassical tradition predicts equalisation of factor returns and income convergence through spatial reallocation, while the new economic geography literature shows that local increasing returns and agglomeration externalities can sustain persistent spatial inequality (Fujita et al., 1999; Puga, 1999). Most existing contributions, however, analyse the spatial dynamics of only one factor at a time, leaving the joint behaviour of labour and capital largely unexplored (Desmet and Rossi-Hansberg, 2014; Quah, 2002).

This paper develops a continuous-space, continuous-time model in which both physical capital and labour are mobile across a set of locations arranged on a circle. Workers relocate following utility gradients — net of location-specific housing and congestion costs — while firms reallocate capital in response to spatial differentials in profit rates. Both agents are myopic and face convex mobility costs, giving rise to a gradient-flow structure. Local capital accumulation follows a Solow-type saving rule, and local total factor productivity depends on the density of workers, capturing agglomeration externalities (Ciccone and Hall, 1996). An idiosyncratic component in workers' location preferences provides a micro-founded diffusion term in the labour dynamics (Fiaschi and Ricci, 2022).

The paper's central result is that the joint interaction of these two mobile factors can generate *Turing instability*: a spatially uniform equilibrium that is initially stable becomes unstable as capital accumulates, triggering the spontaneous emergence of persistent spatial clusters — dense agglomerations of labour and capital separated by depopulated areas. The mechanism is driven not by differential diffusion rates, as in the classical biological formulation (Murray, 2007), but by the interplay between capital accumulation and agglomeration forces embedded in the production technology. Formally, a threshold $L^*(K)$, increasing in the local capital stock, separates stable from unstable uniform configurations. Numerical simulations confirm the theoretical predictions and illustrate the characteristic two-phase dynamics: an initial convergence toward uniformity followed by a sharp divergence into Turing spots.

Keywords: spatial economics, Turing instability, factor mobility, agglomeration, capital accumulation, regional convergence.

JEL Classification: R11, R12, O18, C65.

1 Introduction

The geographical distribution of economic activity is one of the most striking and persistent features of market economies. Both across countries and within them, labour and capital are highly concentrated in space, and these concentrations prove remarkably durable over time. Understanding the forces that generate and maintain such spatial heterogeneity is therefore a central challenge for both growth theory and economic geography.

The neoclassical tradition offers a clean answer: since factors are subject to diminishing marginal productivity, regions with relatively scarce labour or capital enjoy higher factor returns, attracting inflows from elsewhere. Under this logic, factor mobility is unambiguously a force for spatial convergence, gradually equalising per-capita incomes across locations (Barro and Sala-i Martin, 2004). The empirical record, however, suggests a considerably more complex picture. While some convergence is observable across rich economies over long horizons, large and persistent disparities remain common, and there is substantial evidence that agglomeration — the tendency of economic activity to cluster in space — is a dominant feature of spatial dynamics (Ciccone and Hall, 1996; Mirza et al., 2021).

The new economic geography (NEG) literature provides an alternative framework, centred on the role of increasing returns, trade costs, and demand linkages (Fujita et al., 1999; Krugman, 1991; Puga, 1999). In NEG models, local increasing returns create cumulative causation: an initial advantage attracts further activity, which reinforces the advantage, leading to a stable core-periphery equilibrium rather than convergence. A particularly compelling illustration is urban economics, where local returns to human capital density create self-reinforcing agglomeration dynamics: higher wages attract skilled workers, whose concentration further raises wages, driving a positive feedback loop. Rising congestion costs and housing rents constitute the counteracting centrifugal force that prevents unlimited concentration (Krugman, 1998; Mossay, 2003; Mossay and Picard, 2011).

Despite this rich body of work, a notable limitation persists: most models in the spatial economics literature analyse the dynamics of only one factor at a time. NEG models focus almost exclusively on the spatial distribution of workers (Krugman, 1991; Xepapadeas and Yannacopoulos, 2016), while capital is typically treated as either perfectly immobile or as an instantaneous residual that adjusts without frictions (Brito, 2004). The possibility that the *joint* movement of labour and capital gives rise to qualitatively new dynamics — dynamics that could not arise with either factor alone — remains largely unexplored.

This paper addresses that gap. We develop a model in continuous space and continuous time in which both physical capital and labour are explicitly mobile, each governed by its own gradient-flow reallocation equation. Workers move in response to utility differentials; firms move capital in response to profit-rate differentials. In both cases, agents are myopic and face quadratic mobility costs, which generate the classical advection structure. Labour dynamics additionally incorporate a diffusion term arising from idiosyncratic heterogeneity in workers' location preferences, following the micro-foundation developed in Fiaschi and Ricci (2022). Capital evolves through both spatial reallocation and local accumulation driven by a fixed saving rate, as in Desmet and Rossi-Hansberg (2014) and the broader tradition of spatially extended Solow models (Brito, 2004; Puu, 1985).

The central finding of the paper is that the interaction between capital accumulation and labour mobility generates what mathematicians call a *Turing instability* (Murray, 2007): a spatially uniform configuration that is initially stable may become unstable as capital accumulates, giving rise to spontaneous pattern formation in the form of spatial clusters. The mechanism is entirely endogenous: as the economy saves and accumulates capital, the threshold $L^*(K)$ — the minimum labour stock required to sustain a stable uniform distribution — rises. When the aggregate population falls below this threshold, any spatial perturbation is amplified rather than dampened, and the economy converges to a spatially heterogeneous equilibrium characterised by a few dense agglomerations separated by depopulated areas. This two-phase dynamics — initial convergence toward uniformity, followed by sudden divergence into persistent clusters — echoes strikingly the observed long-run spatial evolution of populations, as illustrated by the historical record of population density in the United States from 1492 to the present.

Turing-type mechanisms have attracted growing interest in economic modelling. Brock and Xepapadeas (2010) study pattern formation in coupled economic-ecological systems. Aly (2012) examine Turing instability in a stylised capital-labour model. Zincenko et al. (2021) apply Turing analysis to an economic-demographic system. Ohtake (2024) explore pattern formation via advection-diffusion in a new economic geography setting. Friesen et al. (2019) fit a reaction-diffusion model to the morphogenesis of US urban systems. Relative to this literature, our contribution is threefold. First, we embed the Turing mechanism in a fully micro-founded spatial growth model with two mobile factors, competitive factor markets, and endogenous capital accumulation. Second, we show that Turing instability can arise without differential diffusion rates — the classical sufficient condition in mathematical biology (Madzvamuse et al., 2014; Marciniak-Czochra et al., 2017; Murray, 2007) — driven instead by the non-linear interaction between capital accumulation and agglomeration spillovers. Third, we provide an analytical characterisation of the instability threshold and its dependence on structural parameters, yielding clear comparative statics and policy implications.

The paper is organised as follows. Section 2 presents the model. Section 3 analyses the spatial equilibrium and derives the instability threshold under a tractable parametric specification. Section 4 states the formal stability results. Section 5 reports numerical explorations across several parameterisations. Section 6 concludes.

2 The Model

2.1 Setup

Consider an economy populated by a continuum of locations indexed by $x \in \Omega$, where Ω is a one-dimensional compact domain with periodic boundary conditions — equivalently, a circle or torus. Each location is simultaneously inhabited by workers and firms. There are two factors of production, labour $L(x,t)$ and physical capital $K(x,t)$, which denote the densities of workers and capital at location x at time t, respectively. Land is abstracted from; instead, the model explicitly accounts for location-specific congestion costs and housing costs associated with population density.

The spatial domain has no borders: the periodicity condition $L(0,t) = L(L_x,t)$ and $K(0,t) = K(L_x,t)$ ensures that the domain's endpoints are identified, so that no location enjoys a privileged border position. This topological choice eliminates boundary effects and allows us to focus on endogenous spatial patterns rather than effects driven by the geometry of the domain.

2.2 Workers' Utility and Location Choice

A representative worker located at x at time t derives instantaneous utility

$$u(x,t) = w(x,t) + c(x,t), \tag{1}$$

where $w(x,t)$ is the local wage and $c(x,t)$ captures the net cost of living at x. The cost function satisfies

$$c(x,t) = c(w(x,t),\, L(x,t))\,, \tag{2}$$

with $\partial c/\partial w \geq 0$ (housing costs rise with local income, reflecting the positive income elasticity of housing demand) and $\partial c/\partial L \leq 0$ (congestion raises the cost of living as population density increases, reducing net utility). The cost term thus enters (1) negatively: congestion and rents are the centrifugal forces that offset the centripetal pull of higher wages, in the spirit of Mossay (2003) and Mossay and Picard (2011).

In addition to the systematic component, individual utility may incorporate an idiosyncratic component reflecting heterogeneous location preferences. As shown in Fiaschi and Ricci (2022), the aggregation of such micro-level heterogeneity introduces a Fickian diffusion term in the aggregate labour dynamics.

2.3 Workers' Spatial Reallocation

Workers relocate across space in order to maximise utility. Expectations are static and behaviour is myopic: workers treat current utility at each location as permanent. Mobility is costly, with costs that are convex in distance (e.g., quadratic). Under these assumptions, the spatial evolution of the labour distribution is governed by:

$$\partial_t L(x,t) = -\frac{1}{\gamma_L}\,\partial_x[L(x,t)\,\partial_x u(x,t)] + \sigma_L\,\partial_{xx}L(x,t) + n(x,t)\,L(x,t), \tag{3}$$

where $\gamma_L > 0$ is the marginal cost of labour mobility, $\sigma_L \geq 0$ captures the variance of idiosyncratic location preferences, and $n(x,t)$ is the local net population growth rate, reflecting the balance of births, deaths, and exogenous demographic shocks.

Equation (3) has a natural economic interpretation. The first term on the right-hand side is an *advection* term: labour flows toward locations where the spatial gradient of utility $\partial_x u$ is positive. Workers flow out of locations where utility is locally declining and into locations where it is locally increasing. The parameter γ_L governs the speed of adjustment: greater friction reduces the sensitivity of migration to utility differentials. The second term is a *diffusion* term arising from idiosyncratic preference heterogeneity, which tends to smooth out spatial concentrations. The third term captures natural population growth, proportional to the local stock.

2.4 Firms' Spatial Reallocation and Capital Dynamics

Firms allocate their capital across locations by following spatial differentials in the profit rate. Like workers, firms are assumed to have static expectations, behave myopically, and face mobility costs that are convex in distance. The spatial evolution of physical capital is governed by:

$$\partial_t K(x,t) = -\frac{1}{\gamma_K}\,\partial_x[K(x,t)\,\partial_x r(x,t)] + I(x,t) - \delta K(x,t), \tag{4}$$

where $\gamma_K > 0$ is the marginal cost of capital mobility, $r(x,t)$ is the local rate of return on capital, $I(x,t)$ is local gross investment, and $\delta > 0$ is the depreciation rate of physical capital.

The first term in (4) is the spatial advection of capital toward high-return locations. The remaining terms constitute the standard Solow-type local accumulation equation: gross investment less depreciation.

2.5 Production Technology and Local Technological Progress

Output at location x and time t is produced according to:

$$Y(x,t) = A(x,t)\,F(K(x,t),\,L(x,t))\,, \tag{5}$$

where $A(x,t)$ is local total factor productivity and $F(\cdot)$ is a neoclassical production function.

Assumption 1. $F : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ is twice continuously differentiable, strictly increasing and strictly concave in each argument, homogeneous of degree one, and satisfies the Inada conditions: $\lim_{K\to 0} F_K = \lim_{L\to 0} F_L = +\infty$ and $\lim_{K\to\infty} F_K = \lim_{L\to\infty} F_L = 0$.

Local TFP is assumed to depend on the spatial distribution of workers, capturing the agglomeration externalities and knowledge spillovers documented in the urban economics literature (Ciccone and Hall, 1996; Krugman, 1998; Romer, 1990):

$$A(x,t) = g(L(\cdot,\,t))\,, \tag{6}$$

where $g(\cdot)$ is an increasing function of the local labour density, $g' > 0$. The micro-foundations for this specification, based on the local density of knowledge interactions among workers, are developed in Fiaschi and Ricci (2022). Here we treat (6) as a maintained assumption.

2.6 Factor Markets, Investment, and Aggregate Dynamics

Factor and goods markets are assumed to be perfectly competitive and frictionless. The output price is normalised to one at every location. Under these conditions, factors are rewarded according to their marginal products:

$$w(x,t) = \frac{\partial Y(x,t)}{\partial L(x,t)} = g(L(\cdot,t))\ F_L(K(x,t),\ L(x,t))\,, \tag{7}$$

$$r(x,t) = \frac{\partial Y(x,t)}{\partial K(x,t)} = g(L(\cdot,t))\ F_K(K(x,t),\ L(x,t))\,. \tag{8}$$

Investment is financed by local saving. Following the myopic behavioural tradition of the paper, saving obeys a fixed-rate behavioural rule:

$$I(x,t) = S(x,t) = s\,Y(x,t), \tag{9}$$

where $s \in (0,1)$ is the (common) saving rate.

Collecting equations (3)–(9), the aggregate spatial dynamics of the economy is governed by the following system of partial differential equations:

$$\begin{cases} \partial_t L(x,t) = -\dfrac{1}{\gamma_L}\,\partial_x[L(x,t)\,\partial_x(w(x,t)+c(x,t))] + \sigma_L\,\partial_{xx}L(x,t) + n\,L(x,t); \\ \partial_t K(x,t) = -\dfrac{1}{\gamma_K}\,\partial_x[K(x,t)\,\partial_x r(x,t)] + s\,g(L(\cdot,t))\,F(K(x,t),\ L(x,t)) - \delta\,K(x,t); \\ w(x,t) = g(L(\cdot,t))\ F_L(K(x,t),\ L(x,t))\,; \\ r(x,t) = g(L(\cdot,t))\ F_K(K(x,t),\ L(x,t))\,; \\ K(x,0) = K_0(x); \quad L(x,0) = L_0(x); \\ K(0,t) = K(L_x,t); \quad L(0,t) = L(L_x,t). \end{cases} \tag{10}$$

The last pair of conditions imposes the periodic boundary, making the spatial domain a closed circle.

3 Equilibrium Analysis

3.1 Definition of Spatial Equilibrium

Definition 1 (Spatial Equilibrium)**.** A spatial equilibrium is a configuration $(L^*(x),\ K^*(x))$ such that $\partial_t L(x,t) = 0$ and $\partial_t K(x,t) = 0$ for all $x \in \Omega$ and all $t \geq 0$.

At a spatial equilibrium, neither labour nor capital flows between locations: the spatial gradients of utility and the profit rate are identically zero everywhere. A natural candidate for such an equilibrium is the *spatially uniform solution*, in which $L(x,t) = \bar{L}(t)$ and $K(x,t) = \bar{K}(t)$ for all x. In this case, the spatial terms in (10) vanish, and the dynamics reduces to a purely local, Solow-type accumulation process.

It is important to note, however, that a spatially uniform equilibrium does not necessarily imply equal distributions of labour and capital. The presence of location-specific cost-of-living components means that equal utility can arise from different combinations of wages and living costs; similarly, spatially heterogeneous technological spillovers can sustain equal profit rates at unequal factor densities. These possibilities are consistent with Definition 1 and are economically meaningful.

3.2 A Tractable Parametric Framework

To obtain analytical results, we specialise the general model to the following parametric specification:

Assumption 2. The production function, cost of living, and TFP are given by:

$$\begin{cases} F(K(x,t),\, L(x,t)) = K(x,t)^{\alpha}\, L(x,t)^{1-\alpha}; \\ c(w(x,t),\, L(x,t)) = -c_h\, w(x,t) - c_c\, L(x,t)^{\phi}; \\ g(L(\cdot,t)) = \beta\, L(x,t)^{\mu}, \end{cases} \tag{11}$$

where $\alpha \in (0,1)$ is the capital share, $c_h \in (0,1)$ is the share of wages absorbed by housing costs, $c_c > 0$ and $\phi > 0$ parameterise the congestion cost function, and $\beta > 0$ measures the intensity of agglomeration externalities with $\mu \geq 0$.

Under Assumption 2, the production function is Cobb-Douglas with capital share α, and local TFP increases as a power function of local labour density. The cost of living combines a proportional housing cost (linear in the wage) and a convex congestion cost.

3.3 Equilibrium Factor Returns and the Labour Threshold

Under Assumption 2, the equilibrium profit rate and wage at any location are:

$$\bar{r} = \alpha\, \beta\, L^{1-\alpha+\mu}\, K^{\alpha-1}, \tag{12}$$

$$w = (1-\alpha)\, \beta\, K^{\alpha}\, L^{-\alpha+\mu}. \tag{13}$$

The net utility of a representative worker, as a function of the local labour stock alone, is:

$$\bar{u}(L) = C_h\, (1-\alpha)\, \beta\, K^{\alpha}\, L^{-\alpha+\mu} - C_c\, L^{\phi}, \tag{14}$$

where $C_h \equiv 1 - c_h > 0$ and $C_c \equiv c_c > 0$ collect the cost parameters. The first term captures the net wage income (gross wage less housing cost) and is increasing in L when $\mu > \alpha$, reflecting the agglomeration externality. The second term captures the congestion cost and is increasing and convex in L.

Proposition 1. *Under Assumption 2 and the parameter restrictions $\alpha < \mu < \alpha + 1$ and $\phi > 1$, the function $\bar{u}(L)$ in (14) is strictly concave in L and attains a unique maximum at*

$$L^* = \left[\frac{C_h\, \beta\, (1-\alpha)\, (\mu-\alpha)\, K^\alpha}{C_c\, \phi}\right]^{\frac{1}{\alpha+\phi-\mu}}. \tag{15}$$

Moreover, L^* is strictly increasing in the capital stock K.

Proof. Differentiating (14) with respect to L yields

$$\bar{u}'(L) = C_h(1-\alpha)\beta K^\alpha(\mu-\alpha)L^{-\alpha+\mu-1} - C_c\phi L^{\phi-1}.$$

Setting $\bar{u}'(L) = 0$ and solving for L gives (15). Strict concavity follows from $\bar{u}''(L^*) < 0$ under the stated parameter restrictions, which ensure that $-\alpha+\mu-2 < 0$ and $\phi-2 > -1$. The monotonicity of L^* in K is immediate from (15): $L^* \propto K^{\alpha/(\alpha+\phi-\mu)}$, with $\alpha > 0$ and $\alpha+\phi-\mu > 0$ under the stated restrictions. □

The intuition behind Proposition 1 is illustrated in Figure 1. For a given capital stock K, there is a labour density L^* at which net utility is maximised. When local labour density is below L^*, agglomeration benefits outweigh congestion costs, and the net utility schedule is upward-sloping; above L^*, congestion dominates. The critical property for the stability analysis is that L^* rises with K: as capital accumulates, the labour density required to sustain a spatially stable uniform equilibrium increases, potentially pushing the economy below its own stability threshold.

4 Stability Theorems and the Route to Turing Instability

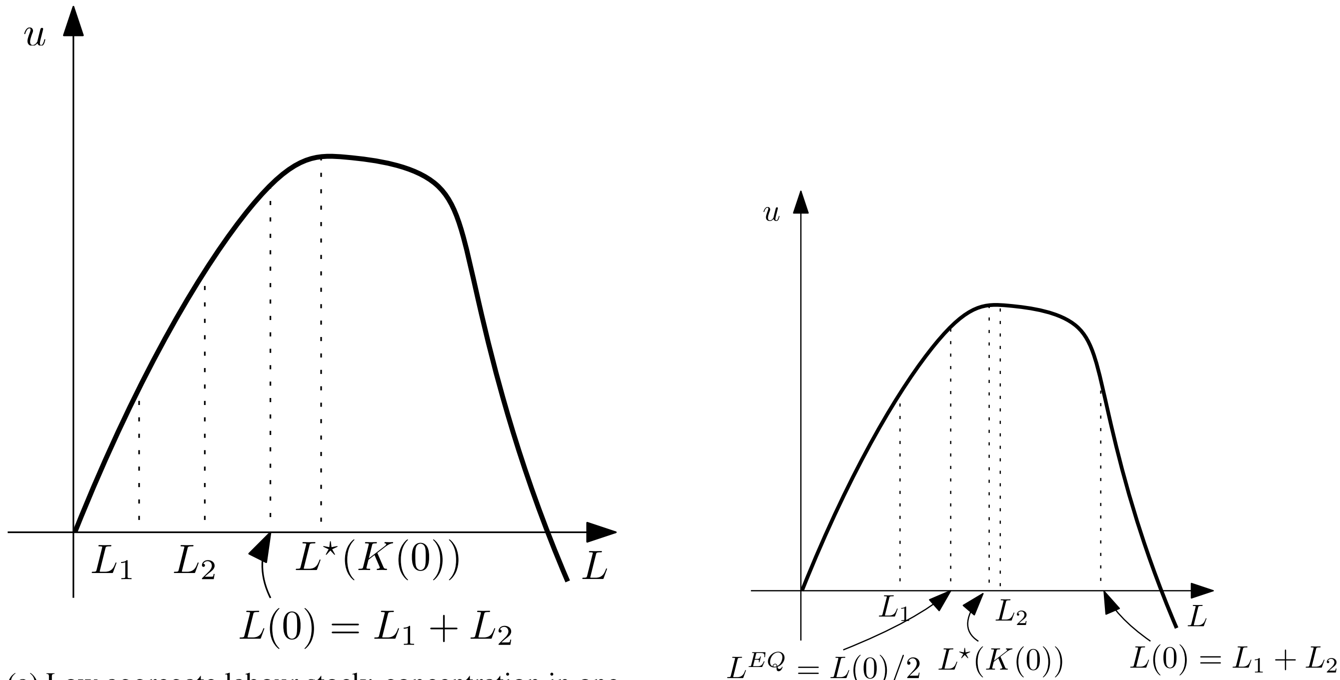
4.1 Formal Stability Results

The first theorem establishes that the spatially uniform solution is self-consistent.

Theorem 1 (Persistence of Uniformity). *If $K_0(x) \equiv \bar{K}$ and $L_0(x) \equiv \bar{L}$ for all $x \in \Omega$, then $L(x,t) = \bar{L}(t)$ and $K(x,t) = \bar{K}(t)$ for all $x \in \Omega$ and all $t \geq 0$. That is, if system (10) starts spatially uniform, it remains spatially uniform for all time.*

Proof. When K_0 and L_0 are spatially constant, all spatial gradients $\partial_x L$, $\partial_x K$, $\partial_x u$, $\partial_x r$ vanish identically at $t = 0$. By the smoothness of F, g, and c, the right-hand sides of the first two equations in (10) also vanish, so the spatial uniformity is preserved instantaneously. By the uniqueness of solutions to (10), the spatially uniform dynamics persists for all $t \geq 0$. □

The second theorem characterises when this uniform solution is stable against spatial perturbations.



(a) Low aggregate labour stock: concentration in one city.

(b) High aggregate labour stock: uniform distribution.

Figure 1: Net utility $\bar{u}(L)$ as a function of local labour density for two levels of aggregate labour stock, with capital held fixed. The threshold L^* separates the region where agglomeration forces dominate (left of the peak) from the region where congestion forces dominate (right of the peak).

Theorem 2 (Stability Threshold and Turing Instability)**.** *Let $L(x,t) = L(t)$ and $K(x,t) = K(t)$ be a spatially uniform solution of system* (10)*. Under Assumption 2 with $\alpha < \mu < \alpha + 1$ and $\phi > 1$, there exists a threshold $L^*(K(t))$, given by* (15)*, such that:*

*(i) If $L(t) > L^*(K(t))$, the uniform solution is **spatially stable**: small perturbations decay and the uniform distribution is restored.*

*(ii) If $L(t) \leq L^*(K(t))$, the uniform solution is **spatially unstable**: small perturbations are amplified, leading to the emergence of spatial heterogeneity.*

Moreover, $L^*(K(t))$ is strictly increasing in $K(t)$.

The proof proceeds via a standard linear stability analysis of system (10) around the spatially uniform steady state. Writing $L(x,t) = L(t) + \varepsilon\, \ell(x,t)$ and $K(x,t) = K(t) + \varepsilon\, k(x,t)$ for small $\varepsilon > 0$, decomposing ℓ and k into Fourier modes $e^{i\kappa x}$ with wavenumber κ, and examining the sign of the real part of the leading eigenvalue of the resulting linearised operator yields the conditions in Theorem 2. The details are deferred to the technical appendix.

4.2 The Mechanism of Turing Instability

The connection between Theorem 2 and Turing instability can be understood intuitively as follows. In the classical Turing framework from mathematical biology (Murray, 2007), pattern formation requires the *activator* species to diffuse more slowly than the *inhibitor*, so that a local perturbation activates a positive feedback that outpaces the restoring diffusion. In our model, the mechanism is different and does not rely on differential diffusion rates (Madzvamuse et al., 2014; Marciniak-Czochra et al., 2017).

Instead, the key driver is the interaction between capital accumulation and the non-monotone relationship between labour density and net utility. When labour density is below the threshold $L^*(K)$, the agglomeration externality is strong enough to make any location with slightly higher density more attractive: workers flow in, raising wages, which attracts more workers. This *local instability* amplifies perturbations. Simultaneously, capital accumulation continuously raises $L^*(K)$, expanding the unstable region. The result is a regime change: an economy that was initially in the stable region (above the threshold) may, as it saves and accumulates capital, enter the unstable region (below the threshold), triggering the formation of spatial clusters.

This constitutes a novel economic route to Turing instability, distinct from those in the existing literature (Aly, 2012; Brock and Xepapadeas, 2010; Zincenko et al., 2021): the instability is not a feature of the long-run steady state but arises *dynamically* as a consequence of the growth process itself. Figure 2 provides a graphical representation of the dynamics in the (L, K) plane, illustrating the division between stable and unstable regions and the trajectory through which capital accumulation drives the economy into the unstable regime.

5 Numerical Explorations

We now illustrate the dynamics of system (10) through a series of numerical simulations. We proceed by progressive layering of complexity, starting from a stripped-down baseline and adding features one at a time. In all simulations, the economy is initialised with a slightly perturbed uniform distribution of both labour and capital.

5.1 Motivating Evidence: Long-Run Spatial Dynamics of Population

Figures 3 and 4 present the historical evolution of population density in the continental United States from 1492 to 2020, and the current cross-sectional distribution, respectively. These figures provide motivating evidence for the paper's central mechanism. In 1492, population was sparse and relatively dispersed. Over the following centuries, the spatial distribution has evolved dramatically: initial concentration along the East Coast, gradual westward expansion, and the eventual emergence of a highly heterogeneous distribution characterised by a small number of very dense metropolitan areas surrounded by vast, sparsely populated territories. This two-phase pattern — initial settlement followed by progressive clustering — is

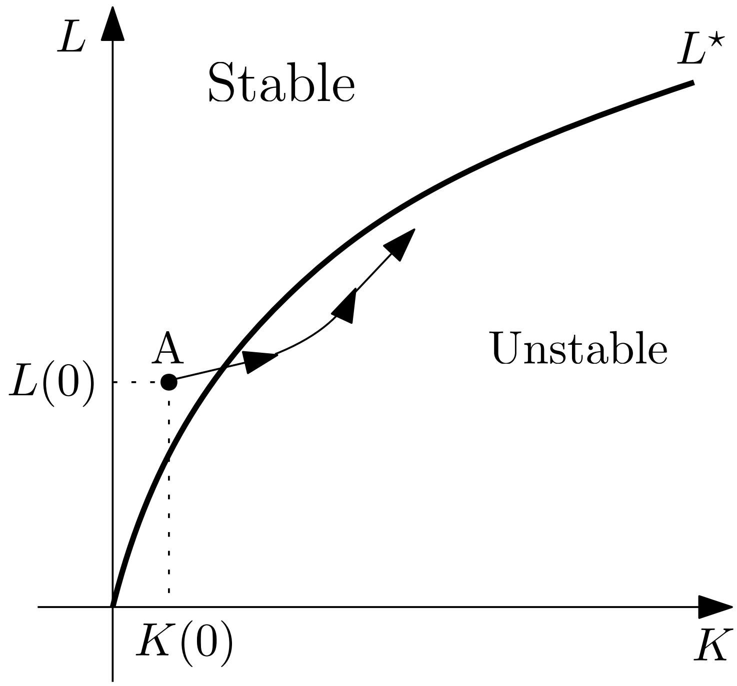


Figure 2: Phase diagram in the (L, K) space. The curve $L^*(K)$ (increasing in K) separates the stable region (above) from the unstable region (below). As the economy accumulates capital along its growth path, the instability threshold rises, and the economy may cross from the stable into the unstable regime, triggering the Turing instability.

qualitatively consistent with the Turing instability dynamics described below (Friesen et al., 2019).

5.2 The Baseline Model Without Accumulation

We first consider a stripped-down version of the model with no factor accumulation and no labour diffusion ($s = \delta = \sigma_L = n = 0$). This *dumb economy* isolates the pure spatial reallocation dynamics. As Theorem 2 predicts, the long-run outcome depends crucially on whether the aggregate labour stock exceeds the threshold $L^*(K)$.

Figure 5 presents a simulation in which the aggregate labour stock exceeds the threshold. The economy converges to the uniform distribution in the long run, confirming the stability prediction.

5.3 Adding Labour Diffusion

The second experiment adds idiosyncratic labour diffusion ($\sigma_L > 0$) while maintaining $s = \delta = n = 0$. Figure 6 shows that the qualitative long-run outcome is unchanged — the economy converges to the uniform distribution — but convergence is substantially accelerated. The diffusion term acts as an additional stabilising force within the stable regime, spreading out any residual spatial heterogeneity more rapidly.

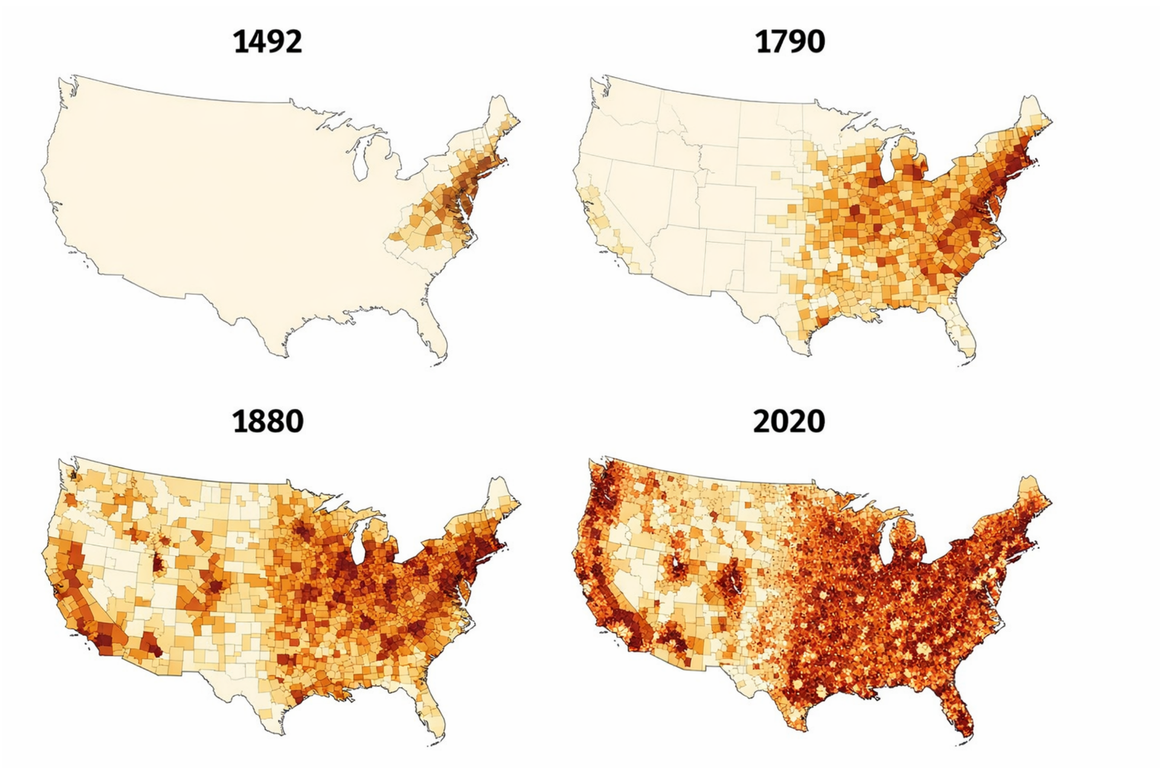


Figure 3: Evolution of population density in the continental United States from 1492 to 2020. Sources: Klein Goldewijk et al. (2011) for the pre-colonial period; IPUMS NHGIS (2023) for 1790–1890; CIESIN (2018) for 1990–2020. The progressive transition from a relatively sparse and uniform distribution toward a highly concentrated, heterogeneous one provides stylised motivation for the paper's main results.

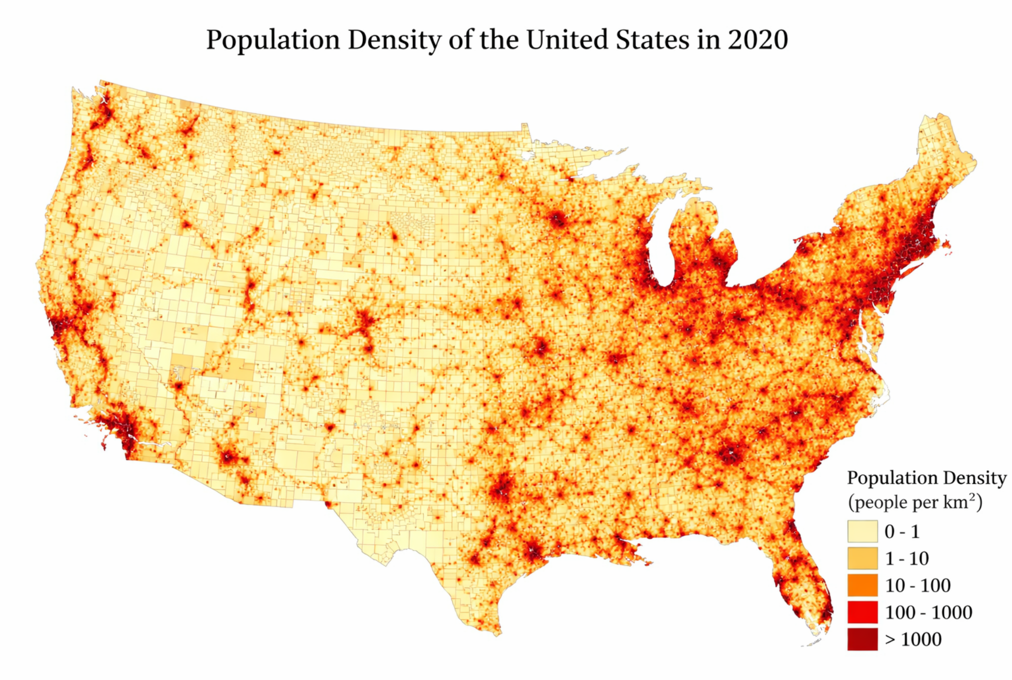


Figure 4: Population density in the United States in 2020 (CIESIN, 2018). The marked spatial heterogeneity, with dense coastal and metropolitan clusters separated by near-empty interior regions, is the empirical counterpart of the Turing spots predicted by the model.

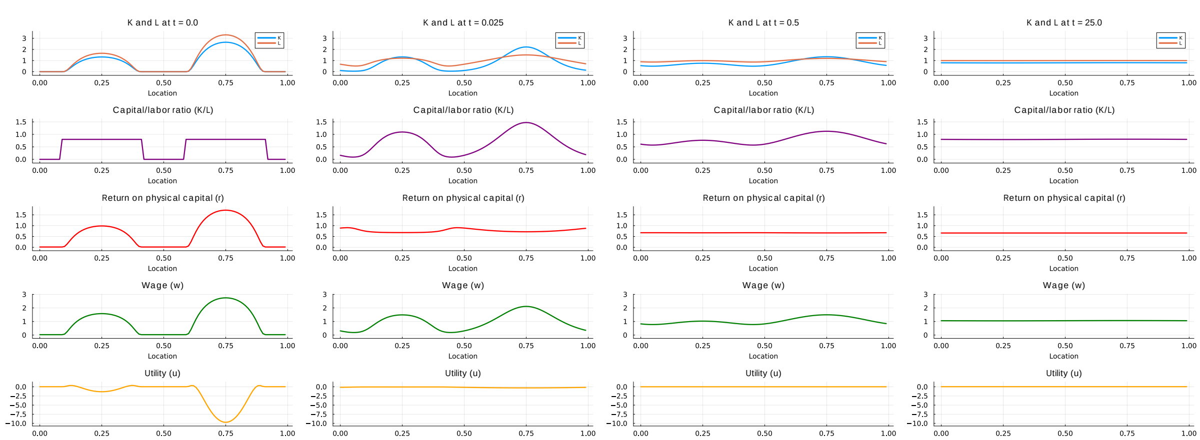


Figure 5: Baseline model without factor accumulation or diffusion ($s = \delta = \sigma_L = n = 0$). Snapshots at $t = 0$, 0.025, 0.5, and 25. Labour stock exceeds the threshold L^*: convergence to the uniform distribution.

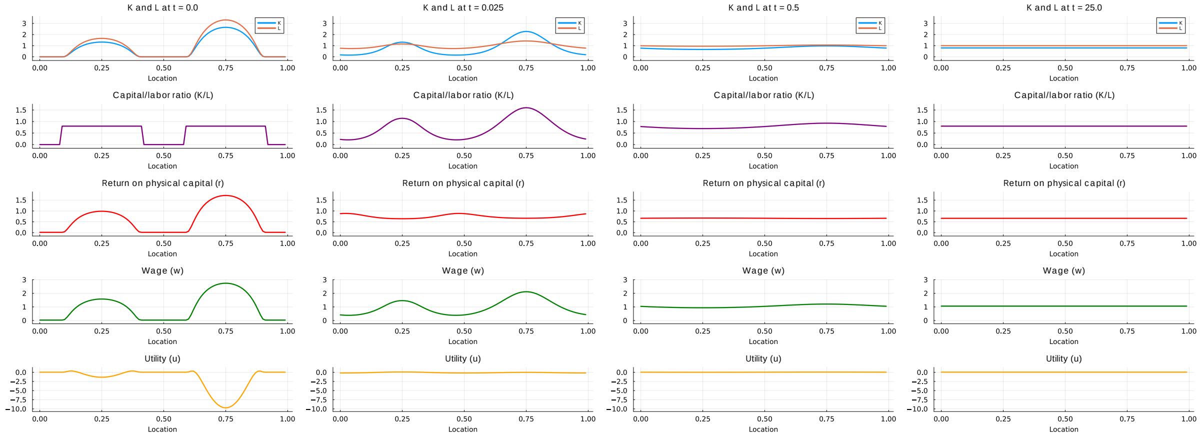


Figure 6: Baseline model with labour diffusion ($s = \delta = n = 0$, $\sigma_L > 0$). Snapshots at $t = 0, 0.025, 0.5$, and 25. Convergence to the uniform distribution is preserved but accelerated relative to Figure 5.

5.4 Population Growth Alone

The third experiment introduces a positive net population growth rate ($n > 0$) together with labour diffusion ($\sigma_L > 0$), while maintaining $s = \delta = 0$. Figure 7 shows that the uniform distribution is recovered in the long run. This result is consistent with Theorem 2: without capital accumulation, the instability threshold $L^*(K)$ does not rise; meanwhile, population growth increases the aggregate labour stock $L(t)$, keeping it above the threshold and maintaining the stable regime.

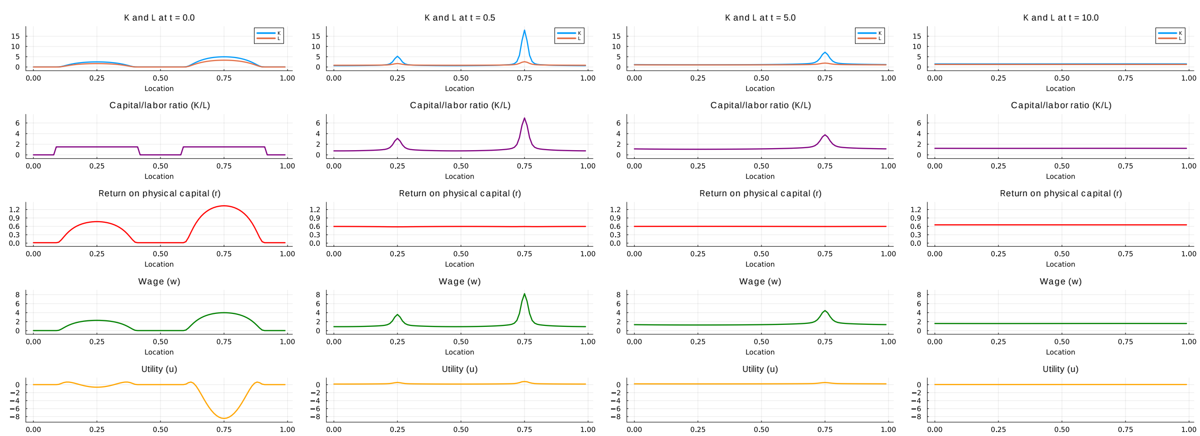


Figure 7: Economy with population growth and diffusion, no capital accumulation ($s = \delta = 0$, $n > 0$, $\sigma_L > 0$). Snapshots at $t = 0, 0.5, 5$, and 10. Uniform distribution is recovered in the long run.

5.5 The Full Model: Turing Instability

The most revealing experiment is the full model with positive saving, depreciation, population growth, and labour diffusion: $s = 0.25$, $\delta = 0.1$, $n = 0.02$, $\sigma_L > 0$. Figure 8 illustrates the dynamics.

The simulation displays the characteristic two-phase pattern of Turing instability. In the early phase (up to approximately $t = 0.5$), the economy converges toward a spatially uniform distribution: the diffusion and gradient-flow terms smooth out the initial perturbations. However, as capital accumulates, the threshold $L^*(K(t))$ rises. At some point, the growing capital stock drives the economy across the stability boundary: $L(t) \leq L^*(K(t))$, and the uniform solution becomes unstable. In the second phase (visible by $t = 5$ and $t = 10$), any residual spatial perturbation is amplified rather than dampened. The economy diverges sharply from the uniform distribution, and persistent Turing spots emerge: a small number of locations attract most of the labour and capital, while other locations become entirely depopulated.

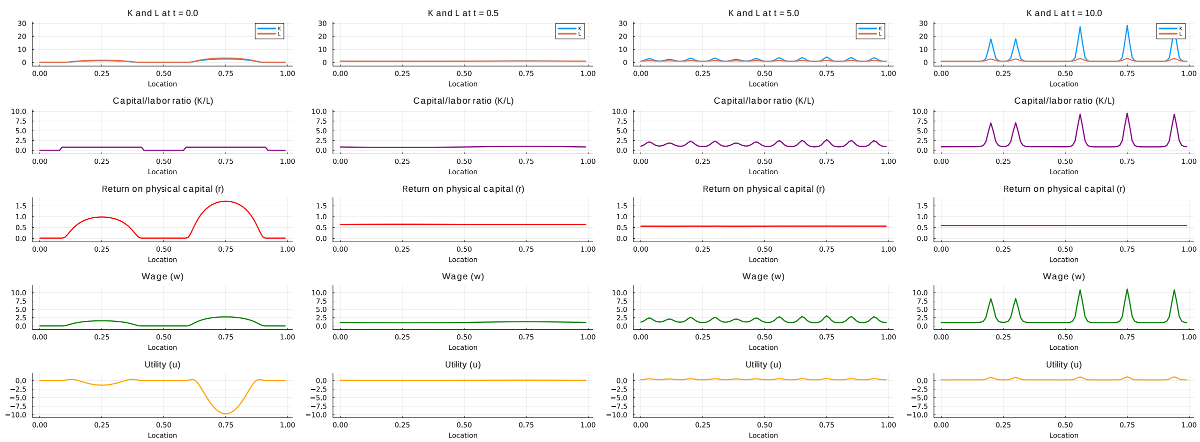


Figure 8: Full model with capital accumulation ($s = 0.25$, $\delta = 0.1$, $n = 0.02$, $\sigma_L > 0$). Snapshots at $t = 0$, 0.5, 5, and 10. The two-phase Turing instability pattern is evident: initial convergence toward uniformity (left two panels), followed by sharp divergence and the emergence of persistent spatial clusters (right two panels), with some areas becoming entirely depopulated.

This result constitutes the core contribution of the paper: Turing instability emerges endogenously from the interaction between capital accumulation and agglomeration forces, without any pre-specified spatial heterogeneity or policy intervention. The economy's own growth process creates the conditions for spatial divergence, offering a novel endogenous explanation for the persistence of regional disparities observed in the data (Fiaschi et al., 2018; Mirza et al., 2021).

6 Conclusion

This paper has developed a continuous-space, continuous-time spatial growth model in which both physical capital and labour are mobile, driven by profit-rate and utility-gradient flows respectively. The joint dynamics of the two factors, coupled with agglomeration externalities and local capital accumulation, can give rise to Turing instability: a spatially uniform equilibrium that is initially stable becomes unstable as the economy grows, triggering the spontaneous formation of persistent spatial clusters.

The key mechanism operates through the interaction between capital accumulation and the non-monotone relationship between labour density and net utility. Agglomeration externalities generate an increasing-returns effect at low densities, while congestion provides a decreasing-returns counterforce at high densities. The balance between these two forces determines a threshold labour density $L^*(K)$

that is increasing in the capital stock. As the economy saves and capital accumulates, this threshold rises, eventually destabilising the uniform equilibrium and generating the Turing pattern. A key property of this mechanism is that it does not require differential diffusion rates between the two factors, in contrast to the standard mathematical biology approach: it is the *growth process* itself that drives the onset of instability.

Several directions for future research are indicated by the analysis. First, a quantitative calibration of the model against regional data — such as the historical US county-level evidence discussed in Section 5 — would provide an empirical assessment of the theory's explanatory power and allow for welfare analysis of the spatial dynamics. Second, the model could be extended to allow for forward-looking capital accumulation and strategic interaction between locations, departing from the myopic assumptions that underlie the gradient-flow structure but potentially generating richer dynamics. Third, the model could be embedded in a two-dimensional spatial domain and equipped with realistic geography, including natural barriers, transport networks, and heterogeneous amenities, to study how the Turing mechanism interacts with these additional sources of spatial heterogeneity.

From a policy perspective, the results carry a cautionary message. Standard policy prescriptions aimed at promoting spatial convergence — such as subsidising factor mobility, reducing congestion, or encouraging population growth — may be insufficient if the capital accumulation process continuously raises the instability threshold. The simulations in Section 5 demonstrate that population growth alone (without capital accumulation) restores the uniform equilibrium; but when both capital and population grow, the race between the rising threshold and the expanding labour stock determines whether the economy converges to uniformity or diverges into clusters. Effective convergence policy may therefore need to directly target the spatial distribution of investment or the intensity of agglomeration externalities in the production technology.

References

Shaban Aly. Spatial inhomogenity due to turing instability in a capital-labour model. *Applied Mathematics*, 3(2):173–195, 2012.

R Barro and Xavier Sala-i Martin. *Economic growth*. Cambridge: The MIT Press, 2004.

Paulo Brito. The dynamics of growth and distribution in a spatially heterogeneous world. 2004.

William Brock and Anastasios Xepapadeas. Pattern formation, spatial externalities and regulation in coupled economic–ecological systems. *Journal of Environmental Economics and Management*, 59(2): 149–164, 2010.

Antonio Ciccone and Robert E Hall. Productivity and the density of economic activity. *American Economic Review*, pages 54–70, 1996.

CIESIN. Gridded population of the world, version 4. *NASA Socioeconomic Data and Applications Center*, 2018.

Klaus Desmet and Esteban Rossi-Hansberg. Spatial development. *American Economic Review*, 104(4): 1211–43, 2014.

Davide Fiaschi and Cristiano Ricci. A microfoundations of spatial agglomeration dynamics. *Mimeo*, 2022.

Davide Fiaschi, Lisa Gianmoena, and Angela Parenti. Spatial club dynamics in european regions. *Regional Science and Urban Economics*, 72:115–130, 2018.

John Friesen, Ruben Tessmann, and Peter Pelz. Reaction-diffusion model describing the morphogenesis of urban systems in the us. In *Proceedings of the 5th International Conference on Geographical Information Systems Theory, Applications and Management*. SciTePress, 2019. doi: 10.5220/0007711300880096.

Masahisa Fujita, Paul Krugman, and Anthony Venables. *The Spatial Economy: Cities, Regions, and International Trade*. MIT Press, 1999.

IPUMS NHGIS. National historical geographic information system. https://www.nhgis.org, 2023.

Kees Klein Goldewijk, Arthur Beusen, and Peter Janssen. The HYDE 3.1 spatially explicit database of human-induced global land-use change. *Global Ecology and Biogeography*, 20(1):73–86, 2011.

Paul Krugman. Increasing returns and economic geography. *Journal of political economy*, 99(3):483–499, 1991.

Paul Krugman. Space: the final frontier. *Journal of Economic perspectives*, 12(2):161–174, 1998.

Anotida Madzvamuse, Hussaini S. Ndakwo, and Raquel Barreira. Cross-diffusion-driven instability for reaction–diffusion systems: analysis and simulations. *Journal of Mathematical Biology*, 70(4):709–743, 2014. doi: 10.1007/s00285-014-0779-6.

Anna Marciniak-Czochra, Grzegorz Karch, and Kanako Suzuki. Instability of turing patterns in reaction–diffusion–ode systems. *Journal of Mathematical Biology*, 74:583–618, 2017. doi: 10.1007/s00285-016-1035-z.

M Usman Mirza, Chi Xu, Bas van Bavel, Egbert H van Nes, and Marten Scheffer. Global inequality remotely sensed. *Proceedings of the National Academy of Sciences*, 118(18), 2021.

Pascal Mossay. Increasing returns and heterogeneity in a spatial economy. *Regional Science and Urban Economics*, 33(4):419–444, 2003.

Pascal Mossay and Pierre M Picard. On spatial equilibria in a social interaction model. *Journal of Economic Theory*, 146(6):2455–2477, 2011.

James D Murray. *Mathematical biology: I. An introduction*, volume 17. Springer Science & Business Media, 2007.

Kensuke Ohtake. Pattern formation by advection-diffusion in new economic geography. *arXiv preprint arXiv:2407.05804*, 2024.

Diego Puga. The rise and fall of regional inequalities. *European Economic Review*, 43(2):303–334, 1999.

Tönu Puu. Continuous spatial modelling in economics. *Papers in Regional Science*, 56:21–36, 1985. ISSN 1056-8190,1435-5957. doi: 10.1111/j.1435-5597.1985.tb00836.x. URL http://doi.org/10.1111/j.1435-5597.1985.tb00836.x.

Danny Quah. Spatial agglomeration dynamics. *American Economic Review*, 92(2):247–252, 2002.

Paul M. Romer. Endogenous technological change. *Journal of Political Economy*, 98(5):71–102, 1990.

Anastasios Xepapadeas and AN Yannacopoulos. Spatial growth with exogenous saving rates. *Journal of Mathematical Economics*, 67:125–137, 2016.

Anna Zincenko, Sergei Petrovskii, Vitaly Volpert, and Malay Banerjee. Turing instability in an economic–demographic dynamical system may lead to pattern formation on a geographical scale. *Journal of the Royal Society Interface*, 18(177), 2021. doi: 10.1098/rsif.2021.0034.