Regional Shape Implications for Equity, Fairness and Sustainability

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Abstract

Local and regional patterns of shape have received much attention and focus because of the implications for equity, fairness and sustainability. A major challenge in regional science has been the measurement and interpretation of shape. This paper examines the theoretical foundations of compactness metrics, and their implications for addressing questions of land use allocation, fair representation in political redistricting, trade area efficiency, environmental support and sustainability more broadly. The roles of geographic information systems and regional analytics are highlighted in the examination of recent legislative districting as well as land use mitigation efforts.

Keywords: Urban form, Redistricting, Spatial Analytics

Introduction

There are many geographic phenomena for which the characterization of compactness is essential due to its desirable (and undesirable) properties. Much attention has been devoted to the shape compactness of political districts in order to ensure that the representation of constituents is fair and equitable, but also that derived districts avoid any attempt to gerrymander in a way that intentionally favors a particular party, racial group or special interest. Compact areas have also proven to be significant in many other ways. Economic efficiency in service provision is generally greatest when the area is compact. Examples include trade regions and school catchments where there are travel costs borne in services obtained and/or offered. The same is true for land use management of vegetation, nature preserves, species monitoring and many others as compact areas are preferred because of their efficacy properties. More sustainable urban areas have been observed to be compact. Physical features like drainage basins are recognized as being compact. Severe weather is often characteristically compact, especially in the formation stage of a tornado. Understanding what is (and is not) compact therefore is a necessity.

The field of geography has played a significant role in recognizing that spatial shape is inherently a function of compactness, with important attempts to quantitatively and mathematically define measures that can best characterize how compact an area is (Ali 2002, O'Sullivan and Unwin 2010, Fan et al. 2015, Beyhan et al. 2020, Murray 2025). Among the early work, Bunge (1962) introduced a measure based on lagged interpoint distances along the boundary of an area. Boyce and Clark (1964) proposed an evaluation of radials emanating from the center of an area, differing from interpoint boundary measures. Massam and Goodchild (1971) suggested the moment of inertia, which is effectively a radial based approach from a center to every point within an area, not just the boundary. Wright et al. (1983) focused on perimeter of an area, concluding that it is most compact when the boundary is a circle. That is, for an equivalent size, an area with the least perimeter is more compact, or more circle like. Inter-person separation detailed by Plane et al. (2019) represents another method to quantify the relative compactness of an area. The intent in mentioning the above work is to emphasize the contributions of geographers, but also that the radial measure continues to figure prominently.

A distinguishing feature of the radial approach of Boyce and Clark (1964) is the explicit focus on deviation from a center to the area boundary, as illustrated in Figure 1. They describe it as "measurement of radial distances from a selected point to the circumference of a geometric form". Substantial radial variability suggests an area is not circular. Further, they observe that the measure can distinguish between different spatial forms with recognized compactness properties, ranging from a circle (lowest radial variability) to a square to a star to a rectangle to a line (most radial variability). As a result of its ability to discern between shapes, there has been much interest in the radial measure of compactness. There are more than one hundred citations to Boyce and Clark (1964) in the literature over the past decade according to Google Scholar, reporting on the use of the radial approach to address urban form, sustainability, service efficiency and many other issues noted above. It has even been relied upon in legal proceedings involving political redistricting (e.g., Virginia House of Delegates v Bethune-Hill et al. in 2014, Supreme Court of the United States #18-281 in 2018). Finally, the radial method is generally available for anyone to apply through a number of opensource libraries, including redistmetrics and redist in R.

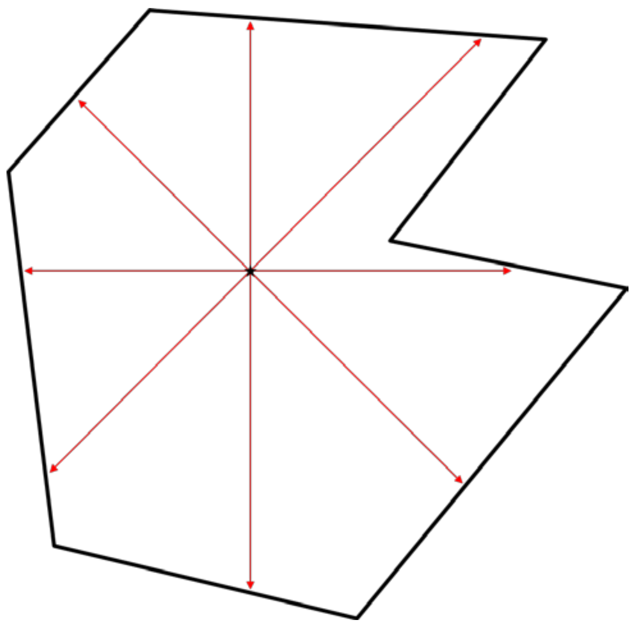


Figure 1. Representative radials emanating from a center extending to the boundary.

While early interest in the radial measure focused on sensitivity issues associated with the number of radii and the center relied upon, little in the way of theoretical properties has been investigated. Research questions that arise include: Why is the radial approach appealing in practice? What is the connection between the intuition of the radial approach and quantitative measurement? Are they any theoretical foundations for the radial approach? How does the radial approach compare to other measures of compactness? This paper investigates these research questions associated with boundary shape compactness in order to comparatively assess performance and properties. The next section offers a review of relevant spatial compactness literature. This is followed by mathematical specification of the radial approach to measure shape compactness. Theoretical underpinnings are then explored. Applications of the radial approach to recognized spatial shapes and political districts are investigated. The paper ends with discussion and concluding comments.

Background

Compactness has been recognized as an important attribute and defining characteristic of the shape of an area (or region, polygon, object, etc.). Further, the measurement of shape compactness has been explored by many, with a host of approaches proposed for quantification from a spatial perspective within geography and allied disciplines. Fairly comprehensive reviews can be found by Frolov (1975), MacEachren (1985), Angel et al. (2010) and more recently Murray (2025). Frolov (1975) attributes an early measure of compactness to Ritter in 1822, but Murray (2025) notes that approaches actually date to late 500 BC or earlier. While compactness has an interesting history of method specification, connections and scientific advances, the review that follows focuses on efforts to quantify the degree to which a spatial shape is compact.

As noted previously, Bunge (1962) in many ways was a geographical shape pioneer in the sense of acknowledging the need for a measure, as had others, but then proposing an approach that attempts to assess how compact an area is. This was done by examining the vertices of a polygon (area object), and in particular deriving distance between them in a lagged fashion. The idea of a lag reflected a systematic skipping of consecutive vertices. Boyce and Clark (1964) avoid criticism of lagged inter-vertex distances, but instead present their own approach that assumes a center from which radials are derived that extend to the area boundary. Their approach then uses a systematic sampling of radials to assess compactness in terms of deviation from an expected value.

Haggett (1966) observes circles as being most economical relative to shape and energy criteria. As for measures, Haggett (1966) discusses work by others that incorporate the radius of the minimum enclosing circle (outer), the maximum inscribed circle (inner) and the longest axis of an area. These measures are more fully organized and formalized in Haggett et al. (1977). Lee and Sallee (1970) advocate and apply a measure of compactness based on symmetric difference of an area to a standard shape.

Massam and Goodchild (1971) proposed a moment of inertia based shape measure, though Murray (2025) notes that others too detailed a similar approach. The essence of the approach is assessment of radial dispersion within an entire area, including the boundary and interior, and comparing this to what would be expected for an equivalent sized circle. Li et al. (2013) offer a contemporary take on moment of inertia and how GIS (geographic information systems) contributes to capabilities for easy computation. Fan et al. (2015) extend moment of inertia to account for population distribution. Differing somewhat is the perimeter based focus of Wright et al. (1983), observing that for any equally sized areas, that with the shortest perimeter would be more like a circle and therefore most compact. Austin (1984) details compactness measures that account for both perimeter and area, though again Murray (2025) reviews work that significantly predates this. Of note, however, is the recognition in Austin (1984) of different types of measures, including perimeter over area, area over diameter, radial, moment of inertia as well as set theoretic and harmonic. Medda et al. (1998) suggest a profile of radii distances. Contrasting somewhat from these methods of compactness, Plane et al. (2019) proposed an inter-person distance approach based on the distribution of people across an area.

The variety of shape compactness measures has been viewed in more reflective ways, with categorization, stratification and summary by many. Frolov (1975) is an early survey that identifies eight different types. As implied above, Austin (1984) also is review oriented. An empirical comparison was undertaken in MacEachren (1985), recognizing the following compactness categories: perimeter and area, circle related, comparison to shape and dispersion about a central point. These categories are fairly reflective of the range of potential approaches, but the comparative evaluation did not include some of the more commonly relied upon measures. Murray (2025) evaluated representative approaches from each of these categories, including isoperimetric quotient commonly referred to as Polsby-Popper (perimeter and area category), Reock (circle related category), moment of inertia (dispersion about a central point category), Schumm (circle related category) and convex hull (comparison to shape category). Collectively, it is fair to say that there remains no consensus on the best measure of shape compactness.

However, MacEachren (1985) does conclude that the radial measure of Boyce and Clark (1964) was highly correlated with other measures, but there was no assessment of Reock or convex hull.

Broad reliance on and assessment of the radial measure of compactness certainly supports further investigation and exploration, prompting the research questions raised in the introduction. There is much beyond political districting concerns for which the use of the radial measure can be noted. Consider the following recent work. Footbridge accessibility was evaluated by Li et al. (2021). Liu and Li (2025) considered city sustainability, finding positive effects on firms, greater efficiency, agglomeration and knowledge spillover, and more innovation when cities are more compact. Wang et al. (2021) evaluated dynamic changes and directions in spatial structure associated with urban expansion. Wu et al. (2023) used the radial approach to detect buildings that disrupt surface airflow. Assessment of urban form was carried out in Su et al. (2021). Medical diagnosis to detect irregularity in thyroid nodule margins was reported in Liu et al. (2024). The radial metric was used to assess multiaxial warp-knitted fabric quality in Dong and Jiang (2025). Of course there is much more, but this highlights the continued significance and popularity of the radial measure.

Critical assessment of radial dispersion has been limited. Cerny (1975) raised issues with the sensitivity of the measure to the utilized center point as well as the need for a sufficient number of radials. Griffith et al. (1986) add observations regarding implicit assumptions, including that the center is within the area, the area is contiguous and that additional attributes cannot be is considered, such as population. The potential for further mathematical insights that may address popularity and appeal, intuition and quantitative measurement, and theoretical foundations make further research on the radial approach salient and timely.

Radial Measure of Compactness

The radial approach for examining compactness introduced by Boyce and Clark (1964) is a measure of dispersion from a center of area o to its boundary based on comparison to an expected compact form. Formal details of this approach rely on the following notation:

j = index of points defining a polygon boundary
$o = \{(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_{m-1}, y_{m-1}), (x_m, y_m)\}$
(x_c, y_c) = center (or central location) of polygon o
$v_j = \overrightarrow{(x_c, y_c)(x_j, y_j)}$

Any area is invariably represented in a digital environment as a polygon, often in the context of GIS. By convention, o indicates the defining coordinates of vertices of the polygon, with consecutive vertices (x_j, y_j) and (x_{j+1}, y_{j+1}) connected by a straight-line segment. Any location i on the boundary has a specific location (x_i, y_i) for which a radial v_i may be defined. A major property of a radial is distance, defined for v_i as follows:

$$r_i = |v_i| = \sqrt{(x_c - x_i)^2 + (y_c - y_i)^2} \quad (1)$$

The primary assumption is that shape compactness is best characterized by the spatial footprint of its boundary. Given this, the radial measure as put forth by Boyce and Clark (1964) is:

$$\sum_{i=1}^{n} \left| \left(100 \frac{r_i}{\sum_i r_i} - \frac{100}{n} \right) \right| \quad (2)$$

where n represents the number of radial end points on the boundary originating from a center (x_c, y_c). Boyce and Clark (1964) suggest at least 16 regularly dispersed radii, (1), but modern implementation using GIS can readily include hundreds, thousands or more.

The radial measure is not particularly straightforward to interpret as structured in (2), but the range of potential values has been established as [0, 200] (Cerny 1975, Austin 1984, MacEachren 1985). Algebraic manipulation offers insight on the underpinnings of the measure. Pulling out the term ${}^{100}\!/_{\sum_i r_i}$ from the summation gives:

$$\frac{100}{\sum_i r_i} \sum_{i=1}^{n} \left| r_i - \frac{\sum_i r_i}{n} \right| \tag{3}$$

The average radii value can be stated as:

$$\bar{r} = \frac{1}{n} \sum_{i=1}^{n} r_i \tag{4}$$

Based on (3) and (4), further simplification gives:

$$\frac{100}{\sum_i r_i} \sum_{i=1}^{n} |r_i - \bar{r}| \tag{5}$$

The average, (4), can also be observed outside the summation of (5), the result of which is:

$$\frac{100}{n\,\bar{r}} \sum_{i=1}^{n} |r_i - \bar{r}| \tag{6}$$

The purpose of this algebraic manipulation and simplification of sorts is to demonstrate that the radial measure in (2) is actually a function of the absolute deviation from the mean observed value, $\bar{r}$. This measure is multiplied by ${}^{100}\!/_{\sum_i r_i}$ in (5) or equivalently ${}^{100}\!/_{n\,\bar{r}}$ in (6). Again, Boyce and Clark (1964) used 16 radials, but others have discussed the need for a larger sample in practice. While contemporary application could readily utilize substantially more radials, many appear to rely on 24 to 60 in practice, rather modest considering the ease with which this can be done in practice. The R packages that implement this assume and/or default to 16 radials (e.g., redistmetrics, redist). The interpretation of this measure is that a value of 0 reflects a perfect circle, while the highest potential value of 200 reflects a straight line (Cerny 1975, Austin 1984, MacEachren 1985). This means that the measure has a range of [0, 200], as noted above, and compactness can be interpreted based on where an observed value lies in this range.

Theoretical Context

There are many texts devoted to statistics, spatial analysis and GIS, the relevance of which is that different measures, metrics, tests and approaches have been developed to better understand observed behavior. Prominent work in statistics and spatial statistics includes that of Clark and Hosking (1986) and Rogerson (2019). Spatial analysis and GIS oriented overviews can be found in O'Sullivan and Unwin (2010), Longley et al. (2015) and de Smith et al. (2025), among others. Based on the overviews offered in these texts and others, it is clear that the measurement of deviation is common in statistics and spatial analysis. Often relied upon is variance:

$$\frac{1}{n}\sum_{i=1}^{n}(a_i - \bar{a})^2 \tag{7}$$

where an observed value a_i is compared to its mean $\bar{a}$. It is not unusual to use the square root of (7) in various ways, namely as a normalized variability from the mean:

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}(a_i - \bar{a})^2} \tag{8}$$

This is typically referred to as standard deviation, but also root mean square error or root mean square deviation when comparing observed and predicted values. Many have long advocated the use of mean absolute deviation (or average absolute deviation) instead:

$$\frac{1}{n}\sum_{i=1}^{n}|a_i - \bar{a}| \tag{9}$$

Clark and Hoskings (1986) refer to this as mean deviation around the arithmetic mean. Willmott and Matsuura (2005) conclude that (9) is a superior measure of error and should always be relied upon over standard deviation, (8). In contrast to (3), there is no theoretical range of expected values associated with (9) as it depends on the units of distance measure.

Of course the significance of highlighting mean absolute deviation is the obvious relationship to the simplified version of the radial measure, (6) or (5). The only difference is a scaling coefficient. However, the scaling factor of $1/\bar{r}$ is rather significant in (6) since it serves to standardize mean absolute deviation, making it interpretable.

A geographic information science perspective appears particularly relevant in the context of radial deviation. Beyond the ability to generate, manage, manipulate, analyze and display spatial information, Murray (2018) indicates that two characteristic functions of GIS include buffer and overlay. These functions can be defined in set theoretic terms, making them unambiguous. What will be shown is that absolute radial difference from the mean can be considered in terms of a buffer of distance r around the center location (x_c, y_c). The resulting spatial object when a point is buffered is:

$$\theta = \left\{(x_c, y_c) \subset \mathbb{R}^2 \middle| \sqrt{(x_c - x)^2 + (y_c - y)^2} \leq r\right\} \tag{10}$$

That is, θ represents an object that includes all points (x, y) that are within a distance r of the center (x_c, y_c). This is illustrated in Figure 2, reflecting a circle in this case since the input is a single point. Given θ, assessment of radial deviation is theoretically possible.

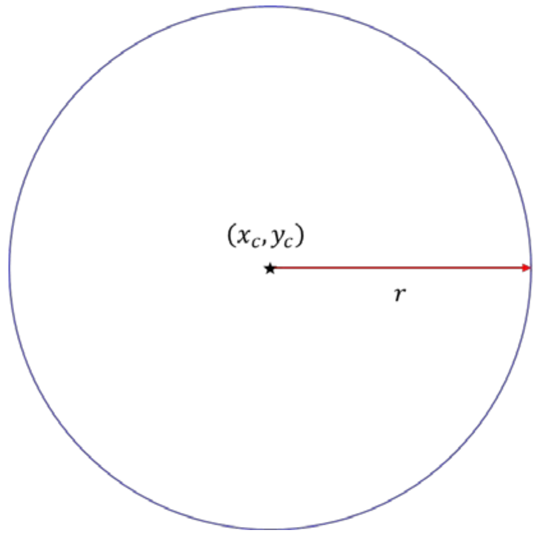


Figure 2. Buffer object created from a center point.

Identification of θ as a geographic object is the first step in assessment. There is also a need to compare θ and o as this is the area being evaluated with respect to radial deviation. In vector GIS terms, overlay is another common spatial analytic function, and specifically enables a range of set based operators to be considered (e.g., intersection, union, difference, symmetrical difference, etc.). Define $\Omega = \{\theta, o\}$ as the set of geographic objects to be evaluated. Vector overlay for this set is:

$$\Psi = \left\{\xi \middle| \kappa \in \wp(\Omega) \text{ where } \forall \xi, \hat{\xi} \subset \kappa \quad \xi \cap \hat{\xi} = \emptyset \ \wedge \ \cup_{\kappa} \xi = \theta \cup o\right\} \quad (11)$$

where $\cap$ is the intersection operator, $\cup$ is the union operator and $\wp(\ )$ is the power set enumerating all combinations of objects, $\{\{\ \ \}, \{\theta\}, \{o\}, \{\theta, o\}\}$, from which overlapping and nonoverlapping components will be comprised. Accordingly, Ψ defined in (11) represents the set of faces (spatial objects) that form the union of Ω set members, $\theta \cup o$, where no two faces $\xi, \hat{\xi} \in \Psi$ overlap, or $\xi \cap \hat{\xi} = \emptyset$. An example of overlay and the resulting faces are shown in Figure 3, with polygon o and buffer object θ overlaid. The result in this case is that there are five unique faces, ξ_1, ξ_2, ξ_3, ξ_4 and ξ_5. Of significance is that the faces represent unique elements and conditions. As an example, $o \cap \theta = \{\xi_3\}$, $o - \theta = \{\xi_1, \xi_2, \xi_5\}$ and $\theta - o = \{\xi_4\}$, with $\cap$ being the intersection operator and – the difference operator.

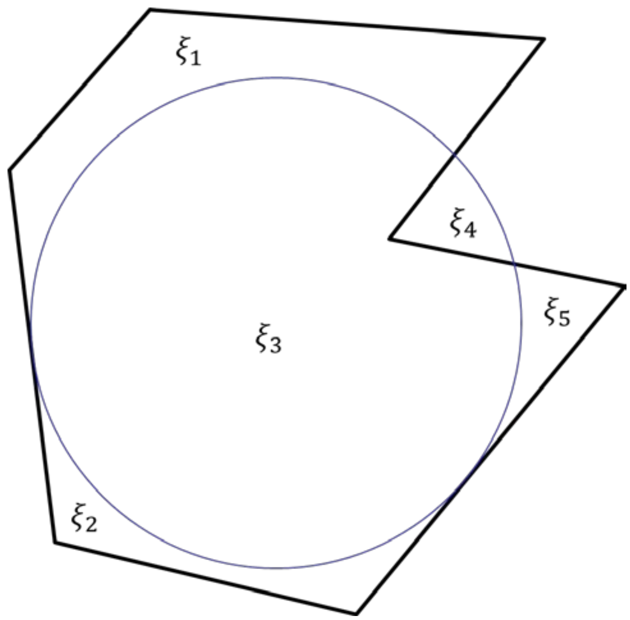


Figure 3. Overlay of θ and o, resulting in unique individual faces ξ.

Bringing the theoretical discussion back to radials of an area o, the intent of Boyce and Clark (1964) was to sample center to boundary distances in order assess associated variability. Too much variability suggests a non-compact area. Limited computational capabilities at that time meant the sample had to be relatively small. However, of interest is what happens when the sample size approaches infinity, $n \rightarrow \infty$. Medda et al. (1998) observe that such asymptotic behavior would eliminate approximation error. To accomplish this, of course, integration along the boundary of polygon o is necessary. Consider a single segment of o, $\overline{(x_j, y_j)(x_{j+1}, y_{j+1})}$, relative to center (x_c, y_c). The average radial distance for this segment would be:

$$\frac{1}{\sqrt{\left(x_j - x_{j+1}\right)^2 + \left(y_j - y_{j+1}\right)^2}} \int_0^1 \sqrt{\begin{array}{c} \left(x_c - \left[w x_j + (1-w) x_{j+1}\right]\right)^2 \\ + \left(y_c - \left[w y_j + (1-w) y_{j+1}\right]\right)^2 \end{array}} \, dw \qquad (12)$$

Examination across all polygon segments would make it possible to derive the actual mean, $\bar{r}^*$, not just an estimate $\bar{r}$.

Given the area of interest o, the actual mean radial distance $\bar{r}^*$ can then be used to derive the buffer object θ that represents the expected radial expanse if the area is circle like. The two objects can then be overlaid to determine the absolute deviation from the mean. This suggests a GIS based approach to absolute radial deviation in the assessment of relative compactness. In fact, Lee and Sallee (1970) proposed evaluation along these lines, referring to it as symmetric difference. This involved comparison of a known shape(s) to the area being considered. In their case, any shape may be of potential interest. They examined a circle, square, triangle and rectangle. Left to be decided by an analyst is how best to size and place the spatial object of comparison. Nevertheless, the symmetric difference measure proposed by Lee and Sallee (1970) in the present context is:

$$1 - \frac{A_{o\cap\theta}}{A_{o\cup\theta}} \tag{13}$$

where $A_{o\cap\theta}$ is the area of the intersection of o and θ and $A_{o\cup\theta}$ is the area of the union of o and θ.[1] Since $A_{o\cup\theta} \geq A_{o\cap\theta}$, this measure will always be in the range [0, 1], with 0 being the most compact and 1 the least compact.

Illustrated in Figure 4a are two spatial objects. The symmetric difference is depicted in Figure 4b. Given this, the measure in (13) reflects the area proportion of highlighted faces. Lee and Sallee (1970) suggested that θ should be sized and positioned so that the area of "nonfit" is minimized, but offered no method(s) for accomplishing this. Subsequent work applying the so-called Lee-Sallee index has been considerable, particularly associated with urban growth simulation where an extension of (13) was proposed and applied initially by Clarke and Gaydos (1998). Worth noting as well is that

[1] The area of a regular polygon can be derived using the defining vertices by $A = \frac{1}{2}\sum_{i=1}^{n-1}(x_{i+1} - x_i)\,(y_{i+1} + y_i)$

Angel et al. (2010) describe a dispersion index as the ratio of the area of o in θ over the area of θ, or ${A_{o\cap\theta}}/{A_\theta}$ in this context. Of course, this differs from (13).

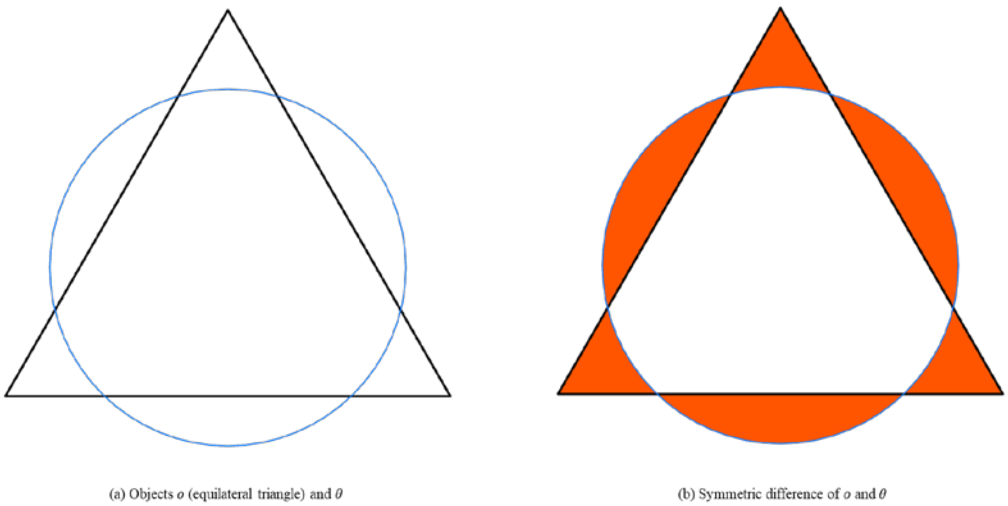


Figure 4. The symmetric difference outcome in overlay (triangle and circle).

Given the potential to eliminate any estimation error, steps are outlined to carry out radial deviation analysis. An algorithm for determining compactness that bridges Boyce and Clark (1964) and Lee and Sallee (1970) based on the theoretical connections above is the following, referred to simply as LSSD($\bar{r}^*$) as an acronym for Lee-Sallee symmetric difference based on the use of $\bar{r}^*$:

1. Identify center of polygon o, (x_c, y_c) (often centroid, but could be central business district or other location)
2. Derive average radial distance $\bar{r}^*$ from center (x_c, y_c) to boundary of o based on (12)
3. Obtain θ using buffer operation (10) applied to (x_c, y_c) with radius $\bar{r}^*$
4. Apply overlay operation (11) to θ and o, enabling determination of symmetric difference (13).

When the polygon is convex with respect to radials from the center point to the boundary, the portions of o outside of θ and the portions of θ outside of o (symmetric difference) represent total absolute radial deviation. As indicated above, Lee and Sallee (1970) opted to standardize the total deviation by the size of the object, $o \cup \theta$, in (13).

Empirical Evidence

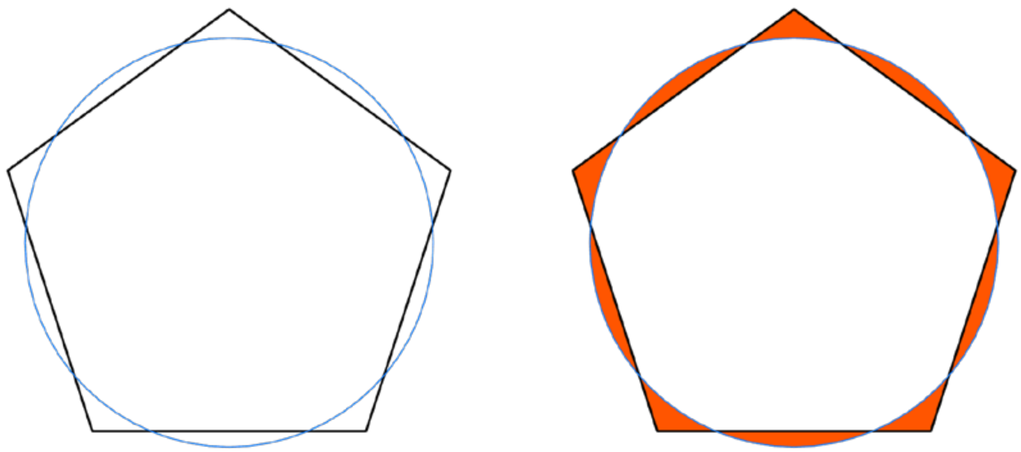
Compactness was assessed for many different observed shapes. The detailed measures were implemented in Python and run on a Windows 11 64-bit laptop (Intel i9-10885H CPU, 2.40GHz with 32 GB). The initial analysis considers 28 shapes reported in the literature (e.g., Boyce and Clark 1964, Griffith et al. 1986, Polsby and Popper 1991, Su et al. 2021) as well as an equilateral triangle, regular pentagon and line area. The shapes from the literature are shown in the Appendix. A summary of four different measures for each geographic shape is given in Table 1. The measures are referred to as: Boyce-Clark, MAD (mean absolute deviation), LSSD($\bar{r}^*$) and LSSD($\bar{r}$). Boyce-Clark in (3) using a radial every degree, which means there will be 360 radials unless there are center to boundary shape non-convexities. Such non-convexities result in more than 360 radials depending on the number of boundary intersections encountered along an individual radial degree. Mean absolute deviation (MAD) in (9) is based on the Boyce-Clark radial sample. LSSD($\bar{r}^*$) is the symmetric difference in (13) proposed by Lee and Sallee (1970) using θ with $\bar{r}^*$ as the buffer radius. Finally, the LSSD($\bar{r}$) measure also is based on (13) but $\bar{r}$ is used instead for creating θ. This last variant is included since in general $\bar{r} \neq \bar{r}^*$ as $\bar{r}$ is an estimate of average radial distance. The analysis utilized the centroid as the center unless it was not within the shape, in which case the representative point function in Shapely was used to identify a location within it. Also, in cases where there are multi-part geometries (non-contiguous polygon), the largest polygon is used.

To begin, consider the triangle shape shown in Figure 4. Table 1 indicates a Boyce-Clark value of 17.648 for this triangle, a MAD value of 6.407, a LSSD($\bar{r}^*$) value of 0.330 and a LSSD($\bar{r}$) value of 0.306. Since the Boyce-Clark measure ranges from [0, 200], the observed value of 17.648 suggests it is not a circle but it is relatively compact since it is on the lower end of the range. The MAD of 6.407 is not particularly interpretable since it is just an average deviation distance, and highlights the significance of the Boyce-Clark measure as it includes a relative scaling parameter to facilitate interpretation. The LSSD($\bar{r}^*$) value of 0.330 indicates that more than 30% of the area of the combined triangle and radial circle, $o \cup \theta$, is outside of the intersected area, $o \cap \theta$, representing radial deviation. Given the theoretical range of [0, 1], this is an indication that the triangle is relatively compact. The LSSD($\bar{r}$) value of 0.306 is close to that of LSSD($\bar{r}^*$), and is interpreted similarly. Another assessed shape in Table 1 is the pentagon, and is shown in Figure 5a. It has a Boyce-Clark value of 5.439, a MAD value of 0.472, a LSSD($\bar{r}^*$) value of 0.106 and a LSSD($\bar{r}$) value of 0.104. Boyce-Clark, LSSD($\bar{r}^*$) and LSSD($\bar{r}$) all indicate that the pentagon is much more compact than the triangle, as expected. Observed in Figure 5b is radial deviation error, roughly 10% of the combined area according to both LSSD($\bar{r}^*$) and LSSD($\bar{r}$). On the least compact side, the line object in Table 1 is a rectangle of 1000 x 0.001 units. Boyce -Clark, LSSD($\bar{r}^*$) and LSSD($\bar{r}$) all rate this as the highest value, indicating it is not compact. The most compact shape is Su-1 in Table 1, a circle. Boyce -Clark, LSSD($\bar{r}^*$) and LSSD($\bar{r}$) each rate this as the lowest measure value.

Table 1. Derived measures of compactness.

Shape	**Boyce-Clark, (3)**	**MAD, (9)**	**LSSD($\bar{r}^*$), (13)**	**LSSD($\bar{r}$), (13)**
Triangle (Figure 4a)	17.648	6.407	0.330	0.306
Pentagon (Figure 5a)	5.439	0.472	0.106	0.104
Line (narrow rectangle)	198.877	5.525	1.000	1.000
Su-0 (rectangle horizontal)	33.159	2.972	0.474	0.488
Su-1 (circle)	0.021	0.002	0.002	0.002
Su-2 (square)	8.874	0.793	0.172	0.165
Su-3 (rotated square)	8.875	0.705	0.172	0.165
Su-4 (rotated octagon)	2.028	0.170	0.040	0.040
Su-5 (rectangle)	25.521	2.275	0.384	0.398
Su-6 (4 pt star)	30.395	1.786	0.507	0.471
Su-7 (H)	40.716	3.237	0.451	0.591
Su-8 (medium rectangle)	61.576	2.977	0.749	0.749
Su-9 (narrow rectangle)	81.827	3.095	0.894	0.880
Su-10 (X)	62.081	4.117	0.653	0.707
Su-11 (octagon)	2.196	0.199	0.043	0.043
PP-0	35.327	3.471	0.487	0.508
PP-1	54.753	2.774	0.727	0.724
PP-2 (cross)	43.058	2.376	0.555	0.579
PP-3	63.331	4.615	0.787	0.818
PP-4 (square + offshoot)	22.665	1.830	0.357	0.386
PP-5 (square + offshoot)	33.774	2.900	0.461	0.489
PP-6 (square + offshoot)	46.934	3.749	0.547	0.606
PP-7	8.632	1.009	0.216	0.227
PP-8 (square missing wedge)	87.967	4.132	0.329	0.731
Griffith-0 (fat 4 pt diamond)	11.910	1.128	0.232	0.217

Griffith-1 (stair 4 pt star)	16.640	1.654	0.304	0.292
BC-0 (polygon)	29.941	4.628	0.391	0.417
BC-1 (Milwaukee)	47.269	4.061	0.541	0.608

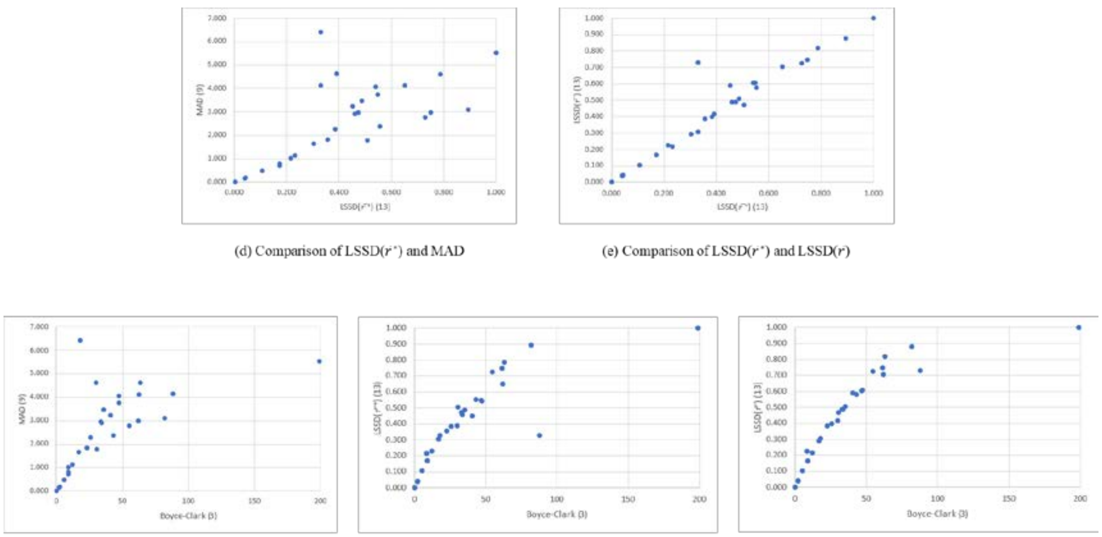


(a) Objects o (regular pentagon) and θ

(b) Symmetric difference of o and θ

Figure 5. Regular pentagon assessment indicating symmetric difference with radial circle.

General trends observed in Table 1 among the compactness measures can be noted. A summary of pairwise linear correlation coefficients (Pearson's) is given in Table 2. As expected, there is a weak correlation between Boyce-Clark and MAD as observed in Figure 6a, with a 0.644 Pearson's correlation coefficient. The comparison of Boyce-Clark to LSSD($\bar{r}^*$) indicates a stronger linear correlation (Figure 6b) of 0.807, but there are a number of significant outliers. The relationship between Boyce-Clark and LSSD($\bar{r}$) is shown in Figure 6c, with a correlation coefficient of 0.848. The MAD and LSSD($\bar{r}^*$) comparison in Figure 6d has a correlation coefficient of 0.695. The strongest linear relationship is observed in Figure 6e (0.955) between LSSD($\bar{r}^*$) and LSSD($\bar{r}$). Collectively, there appears to be much agreement on the relative compactness of the shapes using the Boyce-Clark, LSSD($\bar{r}^*$) and LSSD($\bar{r}$) metrics, but there are clearly a number of outliers where the measures differ significantly. If the line and PP-8 shapes are removed, leaving 26 observations, then the correlation between Boyce-Clark and LSSD($\bar{r}^*$) increases to 0.978. Similarly for Boyce-Clark and LSSD($\bar{r}$) (0.983) as well as LSSD($\bar{r}^*$) and LSSD($\bar{r}$) (0.990). Also included in Table 2 are comparisons with other measures of compactness, including IQ (isoperimetric quotient), S (Schumm), H (convex hull), R (Reock) and MoI (moment of inertia). Technical details for these methods can be found in Frolov (1975), MacEachren (1985), Angel et al. (2010) and Murray (2025), and are summarized in the Appendix. Strong negative correlation among LSSD($\bar{r}^*$) and R (-0.923) and LSSD($\bar{r}^*$) and MoI (-0.913) can be observed. Boyce-Clark correlations are modest overall.



(d) Comparison of LSSD(r^*) and MAD

(e) Comparison of LSSD(r^*) and LSSD(r)

(a) Comparison of Boyce-Clark and MAD

(b) Comparison of Boyce-Clark and LSSD(r^*)

(c) Comparison of Boyce-Clark and LSSD(r)

Figure 6. Scatterplots comparing different compactness measures.

Table 2. Correlation matrix of shape compactness measures reported in Table 1.

	Boyce-Clark	LSSD($\bar{r}^*$)	LSSD($\bar{r}$)	IQ	S	H	R	MoI
Boyce-Clark	1.000							
LSSD($\bar{r}^*$)	0.807	1.000						
LSSD($\bar{r}$)	0.848	0.955	1.000					
IQ	-0.681	-0.831	-0.834	1.000				
S	-0.802	-0.841	-0.731	0.697	1.000			
H	-0.069	-0.350	-0.335	0.552	0.229	1.000		
R	-0.702	-0.923	-0.855	0.774	0.813	0.393	1.000	
MoI	-0.774	-0.913	-0.828	0.766	0.882	0.241	0.822	1.000

Relative assessment using rank order correlation indicates a value of 0.901 between Boyce-Clark and LSSD($\bar{r}^*$), 0.991 between Boyce-Clark and LSSD($\bar{r}$), and 0.927 between LSSD($\bar{r}^*$) and LSSD($\bar{r}$). So in a relative sense there is considerable agreement between the measures. An interesting case of disagreement is for PP-8 in Table 1, with both Boyce-Clark and LSSD($\bar{r}$) finding the shape among the least compact. However, the LSSD($\bar{r}^*$) value of 0.329 suggests that it is more square like than not, perhaps not surprising since it is a square with a missing wedge.

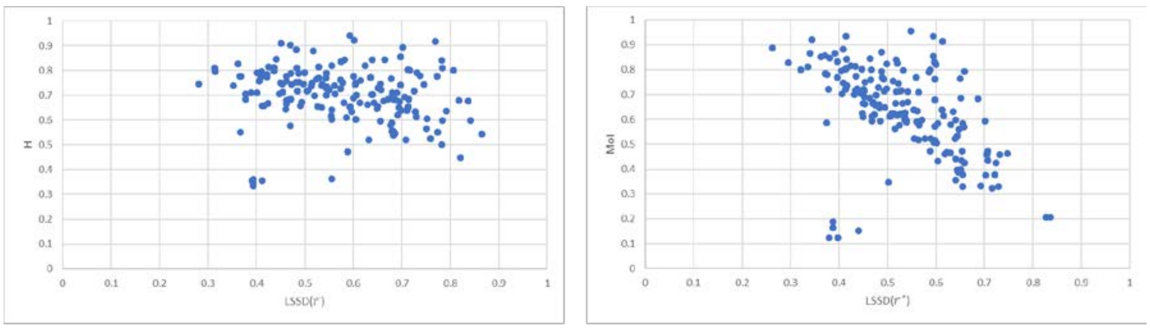
Additional empirical assessment was carried out for California political districts. Evaluated were 2020 U.S. Congressional and State districts obtained from the California State Geoportal (https://gis.data.ca.gov/).[2] There are 52 Congressional districts, 40 State Senate districts and 80 State Assembly districts, for a total of 172 geographical polygons. A matrix summarizing pairwise correlation between observed measurement values is given Table 2. The Boyce-Clark and LSSD($\bar{r}^*$) correlation is 0.733, the correlation between Boyce-Clark and LSSD($\bar{r}$) is 0.939, and the correlation between LSSD($\bar{r}^*$) and LSSD($\bar{r}$) is 0.885. These are somewhat similar to the previous findings, though the Boyce-Clark and LSSD($\bar{r}$) linear relationship is much stronger. Comparing the Boyce-Clark, LSSD($\bar{r}^*$) and LSSD($\bar{r}$) measures with IQ, S, H, R and MoI in Table 3 shows substantially less correlation than observed in Table 2. These measures universally do no correlate well with Boyce-Clark, LSSD($\bar{r}^*$) or LSSD($\bar{r}$). A high of -0.482 is found between LSSD($\bar{r}^*$) and MoI, while a low of -0.125 is found between LSSD($\bar{r}$) and H. Scatterplots of the observed 172 districts in these two cases are given in Figure 7.

Table 3. Correlation matrix of district compactness measures for 2020 California political districts (Congressional, State Senate and State Assembly).

Boyce-Clark	1.000				
LSSD($\bar{r}^*$)	0.733	1.000			
LSSD($\bar{r}$)	0.939	0.885	1.000		
IQ	-0.177	-0.355	-0.258	1.000	

[2] Projected to NAD83 Conus Albers in meters (EPSG 5070).

	Boyce-Clark	LSSD($\bar{r}^*$)	LSSD($\bar{r}$)	IQ	S	H	R	MoI
S	-0.235	-0.413	-0.256	0.480	1.000			
H	-0.109	-0.212	-0.125	0.628	0.729	1.000		
R	-0.246	-0.440	-0.277	0.515	0.991	0.722	1.000	
MoI	-0.283	-0.482	-0.332	0.515	0.930	0.791	0.922	1.000



(a) Comparison of LSSD($\bar{r}$) and H

(b) Comparison of LSSD($\bar{r}^*$) and MoI

Figure 7. Scatterplots of select compactness measures in the assessment of California political districts.

Discussion

The Boyce and Clark (1964) radial deviation measure is appealing for a number of reasons. It is easy to conceptualize given that it examines radials from the shape center to the boundary. There is much intuition behind the approach as well since it seeks to measure deviation from a mean radial value. The paper offered insights on the theoretical foundations of the measure as well, highlighting relationships to mean absolute deviation. Further, buffer and overlay operations in GIS were reviewed as expected radial values can be geographically represented as a circle and compared to a shape of interest. This enables symmetric difference to be determined, equivalent to radial deviation from the mean.

Of particular interest in this paper is how the Boyce-Clark and LSSD($\bar{r}^*$) measures compare since the latter can be considered an actual measure of deviation, not a sample. Generally, these are highly correlated, but important differences can be observed in outlier instances. One is associated with shape PP-8 in Table 1. An aspect of where things differ appears to be associated with the derived radial mean. The sampling approach of Boyce and Clark (1964) indicates $\bar{r}$ =4.697 based on the 698 radials that result. This is far more than 360 that would result if the shape were convex, since one degree angles were used. The actual mean in this case is $\bar{r}^*$ =7.582. The difference is rather stark, and the implications can be observed in Figure 8. The significance in this case seems obvious, as the symmetric difference for the Boyce-Clark approach using $\bar{r}$ =4.697 is 177.843 (shape area is 239.306). In contrast, the LSSD($\bar{r}^*$) approach has a symmetric difference of 82.621. Visual inspection confirms that Boyce-Clark sampling of radii substantially underestimates the mean and as a result 74.32% of the shape is radial deviation from the mean. In contrast, the actual mean ($\bar{r}^*$ =7.582) used in LSSD($\bar{r}^*$) reduced the radial deviation from the mean to 34.4%, representing a much better circle comparison. Across all the shapes and districts evaluated, there are instances where symmetric difference is lower for $\bar{r}^*$, as is the case in Figure 8, but also cases where $\bar{r}$ is lower. This seems to suggest that neither may be an efficient central value for use in absolute deviation. Willmott and Matsuura (2005) do note that the median is more efficient for mean absolute deviation, so it may be advisable to move away from the mean structure in Boyce and Clark (1964). What the best central value should be, however, remains an open research question.

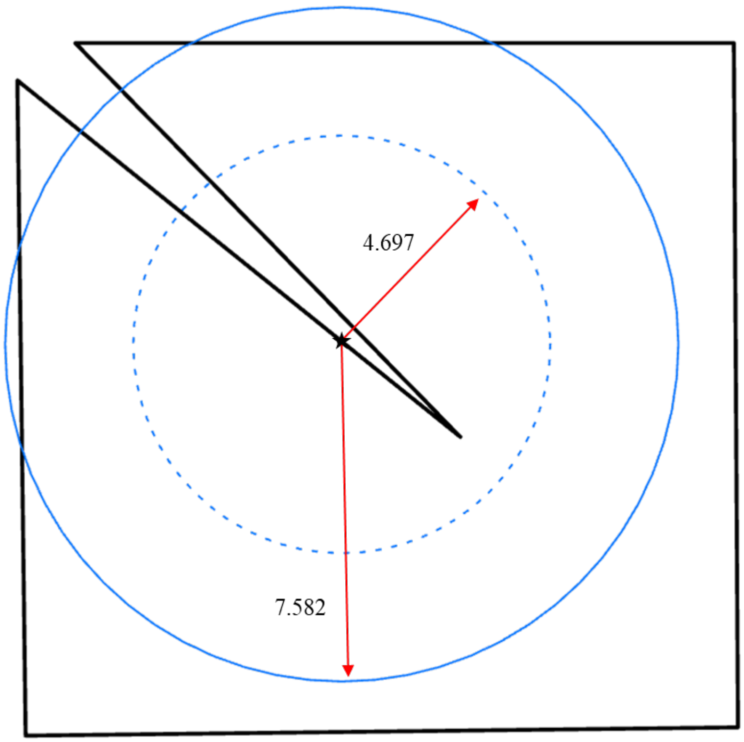


Figure 8. Shape PP-8 with average radial distance circles shown.

The absolute deviation from the mean (or any central value) is easy enough to compute. Further, this may be visualized along the lines shown in Figures 4b and 5b, when convexity between the center to boundary exists. When the shape is not convex, there may be one or more radials that extend to multiple points along the boundary. In such cases, the absolute deviation may not exactly equal the symmetric difference. That is, $\sum_{i=1}^{\infty}|r_i - r| \approx (A_{o-\theta} + A_{\theta-o})$. An open research question is whether such approximation and/or reframing of deviation is acceptable in the context of evaluating compactness.

Many note that the center (x_c, y_c) be inside the polygon, $(x_c, y_c) \subset o$, and often this center is assumed to be the centroid. If the idea is to compare a circle to a shape, which is the central tenet of the Boyce-Clark approach, then the size and location of the circle is something to be determined. Lee and Sallee (1970) indicated this as well, suggesting that symmetric difference, $A_{o-\theta} + A_{\theta-o}$, be as low as possible. That would make this a type of spatial optimization problem. Of course, the central radius value used likely would need further theoretical justification in the context of absolute deviation suggested in Boyce and Clark (1964). These remain open research questions.

The empirical comparison of compactness measures was insightful for many reasons. In general, LSSD($\bar{r}^*$) represents an extension of the Boyce-Clark metric, enabling a move away from sampling to obtain the actual radial mean, $\bar{r}^*$, in comparative assessment. This obviously reduces the potential for bias and error in the application of Boyce-Clark to assess compactness. With respect to relationships between alternative compactness measures, the findings appear to differ somewhat from the conclusions drawn in MacEachren (1985) in their evaluation of 54 county shapes. They found that Boyce-Clark was moderately correlated with IQ (0.75) and S (0.86) but highly correlated with MoI (0.98) and LSSD($\bar{r}$) (1.0). Further, LSSD($\bar{r}$) was moderately correlated with IQ (0.73) and S (0.86) but highly correlated with MoI (0.98). The initial analysis involving the shapes summarized in Table 1 were more or less consistent with this (see Table 2). However, the assessment involving political districts (Table 3) found little correlation when Boyce-Clark or LSSD($\bar{r}$) were compared to IQ, S or MoI (or H or R). In fact, the conclusion appears to be that Boyce-Clark and LSSD($\bar{r}^*$) offer much uniqueness in quantifying compactness that is not readily captured by other metrics. That is, they are particularly valuable approaches, likely explaining the continued use and reliance on Boyce-Clark in practice that can be observed in the literature.

When one considers the use of different compactness measures in a composite score, such as that advocated by Kaufman et al. (2021), Saporito and Maliniak (2022) and others, of radial deviation focused on a shape boundary would not be captured by other measures. The political districts analysis highlights this, suggesting that creating a composite score of different compactness measures needs to be carefully considered. Some measures are correlated and others are not. What remains the case is that no one measure has proven to be superior.

A final point of discussion is the scaling suggested by Lee and Sallee (1970). Indicated in (13) is that they defined the symmetric difference metric as involving the ratio of intersection area, $A_{o\cap\theta}$, over combined area, $A_{o\cup\theta}$. An important research question is whether scaling by $A_{o\cup\theta}$ is theoretically justified. Angel et al. (2010) suggested A_θ instead of $A_{o\cup\theta}$. One might even argue for A_o instead. It appears that a strong case could be made for either, and further that $A_{o\cup\theta}$ may bias the measure in some way since it often inflates the total area without representing a statistical expectation.

Conclusions

Boyce and Clark (1964) is an important and innovate approach to assess compactness, garnering much continued application to study a variety of problems and issues. The emphasis on deviation from the radial mean is inherently geographic. The literature has treated Lee and Salle (1970) as a different type of approach focused on symmetric difference emanating from set theory. However, this paper offered a new perspective that an infinite number of radials actually approaches an area based measure. As a result, it can reflect symmetric difference under certain conditions.

Empirical investigation demonstrated the value of the Boyce-Clark and LSSD measures for assessing compactness. Further, it is evident that these measures encapsulate different characteristics of compactness compared to other measures. While not investigated in this work, a benefit of the symmetric difference approach is that non-contiguous, multi-polygon areas likely are not an issue for a more broadly conceived area approach along the lines of Lee and Sallee (1970), structured and applied as LSSD in this paper using estimated and actual radial means, $\bar{r}$ and $\bar{r}^*$.

The discussion has emphasized that there remain important issues to be addressed in future research. Among these, the following appear particularly important. One is work on the best positioning and sizing of a radial deviation circle. Another involves the interpretation of deviation error, particularly in the context of deviation circle positioning and sizing. Composite integration of compactness measures is something that has emerged due to uncertainty about the best metric to use. Which measures to include and how to integrate them remains ad hoc at best. Finally, the impact and influence of symmetric difference scaling could have profound impacts on the radial deviation measure, and remains poorly understood.

Acknowledgements

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Appendix

Relied upon in the analysis were a number shapes that have been considered and evaluated in the literature. These shapes were digitized for assessment. Figure A-1 depicts the 12 shapes used from Su et al. (2021). Figure A-2 shows the 9 shapes included from Polsby and Popper (1991). Figure A-3 gives the two shapes considered in Griffith et al. (1986). Finally, Figure A-4 includes two shapes relied on in Boyce and Clark (1964).

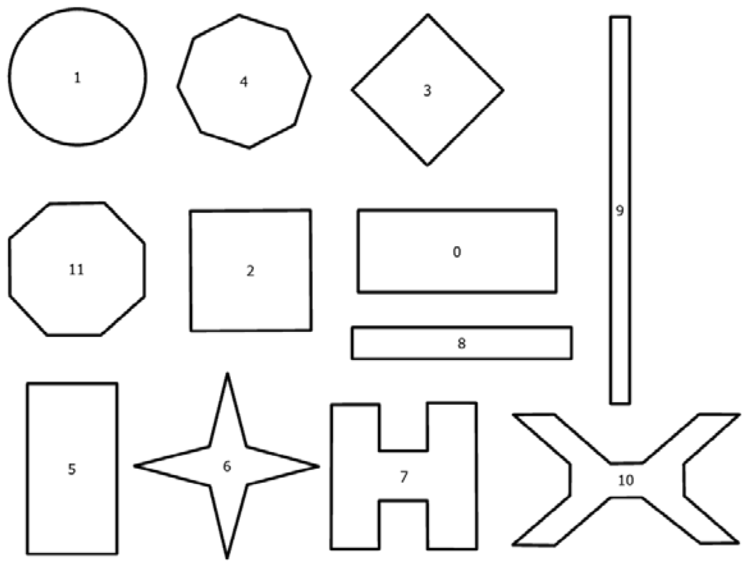

Figure A-1. Shapes based on analysis reported in Su et al. (2021)

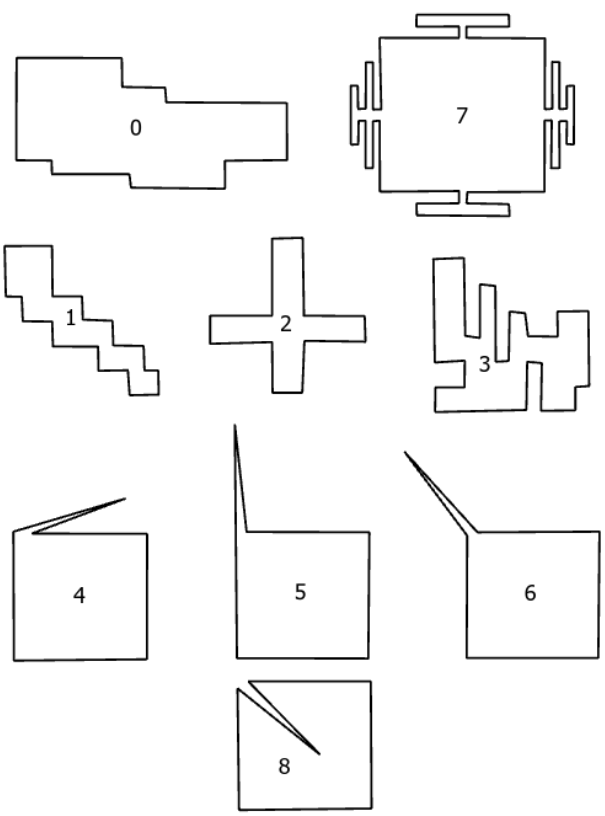

Figure A-2. Shapes based on analysis reported in Polsby and Popper (1991)

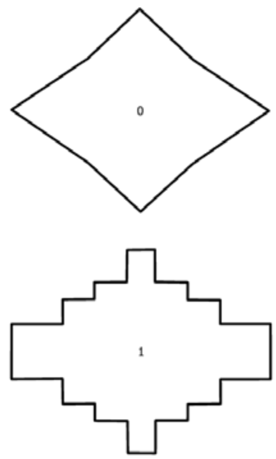

Figure A-3. Shapes based on analysis reported in Griffith et al. (1986)

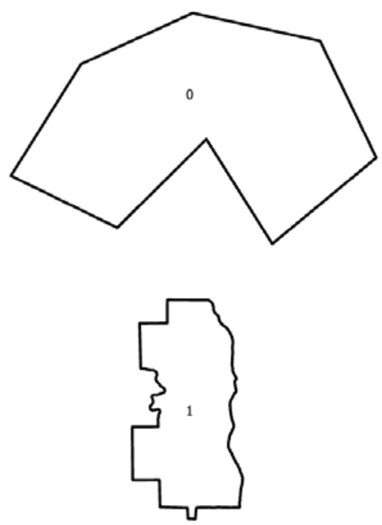

Figure A-4. Shapes based on analysis reported in Boyce and Clark (1964)

In addition to the Boyce-Clark, MAD, LSSD($\bar{r}^*$) and LSSD($\bar{r}$) measures of compactness, the analysis also offers comparison to common metrics relied upon in practice. These measures of compactness include: IQ (isoperimetric quotient), S (Schumm), H (convex hull), R (Reock) and MoI (moment of inertia). Details are offered in Table A-1. Further discussion and specification can be found in Murray (2025), as well as many others.

Table A-1. Commonly applied compactness metrics.

Measure	Specification	Notation
IQ (isoperimetric quotient)	$\frac{4\pi A_o}{p^2}$	p = perimeter of o
S (Schumm)	$\frac{2\sqrt{A_o/\pi}}{d}$	d = maximum diameter of o
H (convex hull)	$\frac{A_o}{A_h}$	A_h = area of convex hull of o
R (Reock)	$\frac{A_o}{\pi r^2}$	r = radius of minimum enclosing circle of o
MoI (moment of inertia)	$\frac{A_o^2}{2\pi I_{(x,y)}}$	$I_{(x,y)} = \int_o \delta^2 \, do$, where δ is the distance from (x, y) to a point in o

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