FRACTAL DIMENSION COMPLEXITY OF GRAVITATIONAL FRACTALS BASED ON TRIANGULAR LATTICES.

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Abstract

The central place theory (CPT) formulated by Walter Christaller in 1933 and widely introduced into the economic geography literature in the 1950s has bcome one of the leading theoretical frameworks describing the spatial order of settlements in the space-economy, and consequently, the geographical differentiation and specialisation of economies in various regions and countries.

In a geometric sense, CPT is essentially based on the possible existence of three types of network configurations that completely cover a plane, i. e., hexagonal, square, and equilateral triangle networks.

The original formulation by Christaller - and in the same spirit by Lösch- as elaborated in many subsequent applications, is dominated by the hexagonal network, while the square and triangular networks are used sporadically (e.g Parr J.(1985), Square Market Areas, Löschian Numbers and Spatial Organization, *Geographical Analysis*, 17(4), 284-301).

From an algebraic point of view, CPT is related to number theory and allows for the determination of new properties of numbers called Löschian numbers (see Kaczorowski, J., Ratajczak, W., Nijkamp, P., Górnisiewicz K., (2024) Economic hierarchical spatial systems – new properties of Löschian numbers, Applied Mathematics and Computation461,128319). However, the basic physical feature of CPT is its stationarity. This means that the hierarchy of central centers results from the "rotation" of market areas, rather than from the social, economic, or cultural properties of these centers that change over time.

In the work by Banaszak, M., Górnisiewicz, K., Nijkamp, P., Ratajczak, W. (2023), on Fractal dimension complexity of gravitation fractals in central place theory. Sci Rep 13, 2343) the authors have shown that urban centers in a hexagonal network may generate irregular areas of attraction (attractors) whose complexity and shape depend primarily on transportation costs. Furthermore, the authors showed that, if transport costs – i.e., the "resistance" of space – are sufficiently high, the areas of attraction of cities located in a hexagon coincide exactly with the shapes and ranges determined by central place theory.

In the present paper, we undertake a similar numerical experiment as the hexagonal form, but now with reference to a triangular grid. The main objective of this work is to determine the relationship between the costs of space resistance occurring in a hexagonal network and the cost implications of resistance characterizing a triangular network. This will allow us to determine at which critical level of transport costs the gravitational fractals transform into regular networks predicted by central place theory. The findings of our research will be illustrated by several visualized computer color simulations.

The spatial structure of a central place system can be analyzed at two distinct but formally connected geometric levels: the configuration of center locations and the partition of space into market areas. These two aspects are governed by different mathematical principles and therefore should be treated separately.

Consider an unbounded isotropic plane $\mathbb{R}^2$ in which centers are located so as to provide uniform coverage of demand. Among all periodic point lattices of given density, the triangular (or hexagonally dense) lattice maximizes the minimum distance between neighbouring centers. Formally, if $\Lambda \subset \mathbb{R}^2$ denotes a lattice of center locations with fixed density, the triangular lattice solves

$$\max_{\Lambda} \min_{\{x,y \in \Lambda, x \neq y\}} d(x,y) \, ,$$

where d is any admissible distance measure. This result explains why triangular arrangements of centers appear naturally in both classical central place models and empirical settlement patterns. Importantly, however, the lattice Λ specifies only the spatial configuration of centers. By itself, it does not determine how surrounding space is allocated among them.

To derive market areas, an explicit rule of spatial assignment is required. In central place theory this rule is based on distance or cost minimization: each location is assigned to the center that can be reached at the lowest cost. Let the distance function be given by

$$d(x,y) = ||x - y||,$$

where ||·|| denotes an arbitrary norm on $\mathbb{R}^2$. The restriction to norm-induced metrics ensures translation invariance, spatial homogeneity, and economic interpretability of distance as generalized access cost. The market area associated with a center $c_i \in \Lambda$ is then defined as the Voronoi cell

$$V(c\_i) = \{x \in \mathbb{R}^2 \colon ||x - c_i|| \ \leq \ ||x - c_j|| \, for \ all \ c_j \neq c_i\}.$$

For any norm, the unit ball $\{x \colon ||x|| \leq 1\}$ is a convex and centrally symmetric set. This property is independent of the particular choice of norm and holds equally for the Euclidean, Manhattan, Chebyshev, and more general Minkowski norms. Because Voronoi cells are constructed as intersections of half-spaces defined by bisectors in the chosen metric, they inherit this central symmetry. In two dimensions, this implies that generic Voronoi cells are convex polygons with an even number of sides, such as hexagons or quadrilaterals. A triangle, lacking central symmetry, cannot arise as a Voronoi cell generated by any norm-based metric.

It follows that a triangular lattice of centers may generate hexagonal or square market areas, depending on the underlying norm, but it cannot generate triangular market areas. The geometry of center placement solves a spacing problem, whereas the geometry of market areas is determined by the structure of the distance metric and the Voronoi construction. Consequently, triangular market areas are incompatible with both classical and generalized formulations of central place theory, unless the fundamental assumptions of metric homogeneity and cost-minimizing spatial behaviour are abandoned.

In our numerical experiment, we decided to build a triangular cluster of central places in the same way that Lösch built hexagonal market areas. For each point of the triangular lattice (x, y), which is a linear combination of the vector base: $(1, 0)$ and $(1/2, \sqrt{3}/2)$, we create a triangle with vertices: $(-x, -y)$, $(-y, x+y)$, $(x+y, -x)$. This triangle is the simplest complement of the Lösch hexagon and has an area proportional to $\frac{3}{2}L = \frac{3}{2}(x^2+xy+y^2)$. (see Figure 1).

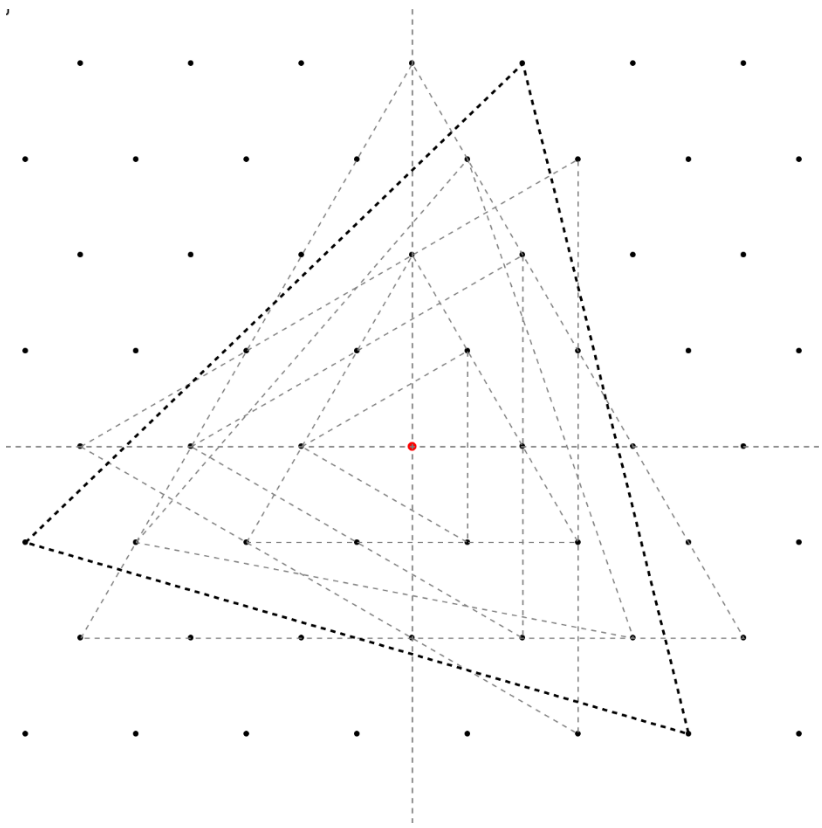

Fig. 1. Triangular market areas of increasing size derived from the following values (x, y):(1,0), (1,1), (2,0), (2,1), (3,0), (2,2), (3,1).

In the next step, a map was created (similar to the work of Banaszak et al., 2023) that illustrates the attractiveness of centers at given points based on Newton's equation of motion:

$$\frac{d^2\vec{r}}{dt^2} = -\nabla U - \mu \frac{d\vec{r}}{dt}.$$

With different values of the coefficient μ friction of space (transport costs), the map is characterized by a different complexity of gravitational attractors of central places. One way to capture these complexities is to calculate the fractal dimension of the boundary of the basins of attraction (gravitational fractals). Due to the numerical nature of the study, it is not possible to determine the exact limit values and a fixed size of the maps (n x n, n=2000) must be determined. Then the ruler fractal dimension can be approximated by the formula:

$$d_r = \frac{\log(P_r)}{\log(n)}.$$

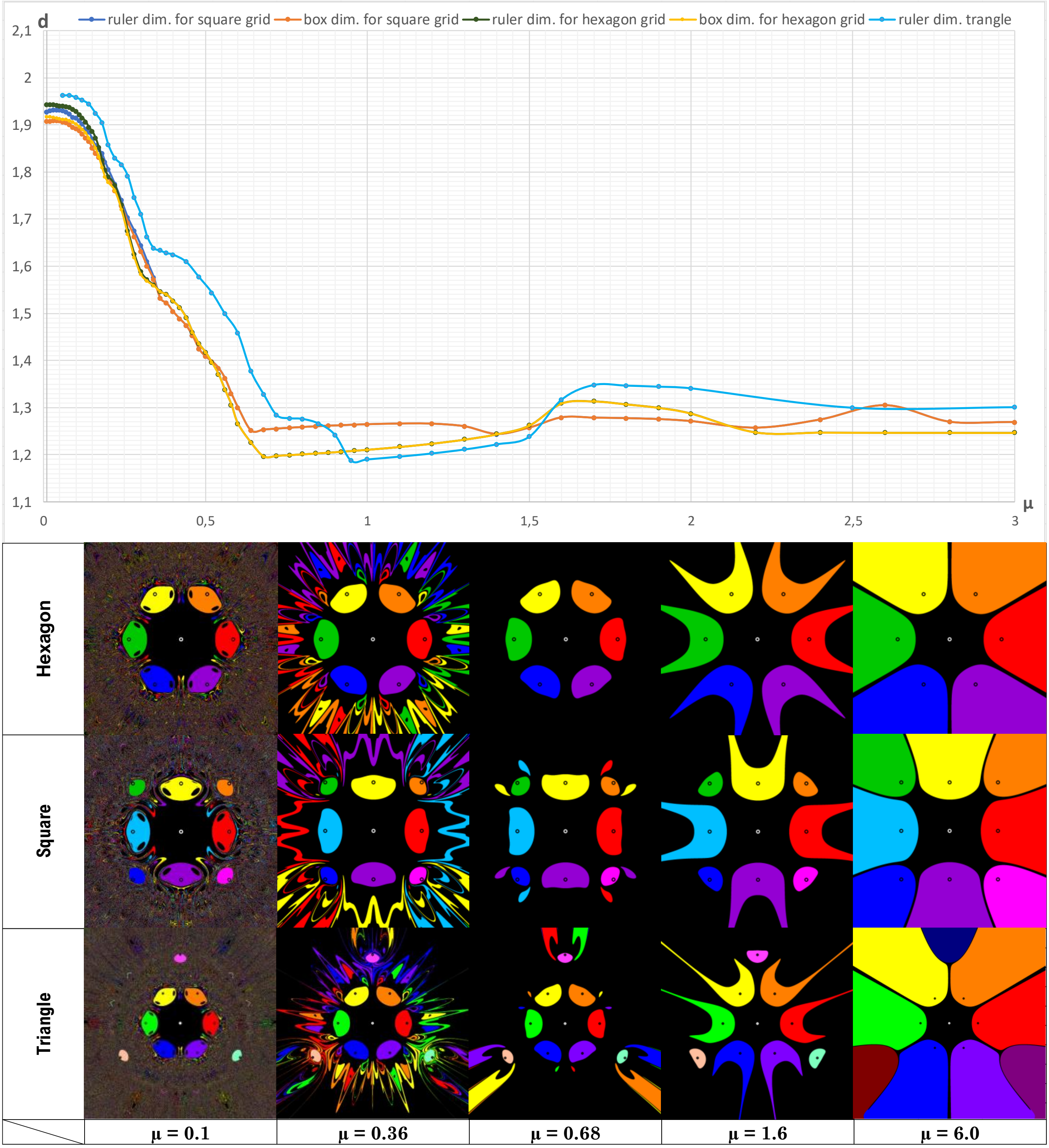

Fig. 2. Distribution of the fractal dimension d of the spatial resistance function μ, for $m = 4$ (hexagon, square) and $m = 4.5$ (triangle).

Hence, it can be stated that although the spatial distribution of central places is **triangular** (or square) in nature, the hierarchical network **of market areas** - areas of influence - has **a hexagonal** (or square) form. That is, a triangular network describes the location of central centers, and a hexagonal network describes the areas of their influence.

It is worth noting that when central centers form a triangular network, then rings of the neighbor layer are formed around each center, which are related to **the triangular numbers**. These numbers are defined as below:

$$T_n = \frac{n(n+1)}{2},$$

where is a non-negative integer. Triangular numbers are closely related to the Lösch numbers L, because the equation

$$L = m^2 + mn + n^2$$

can be saved as:

$$L = T_m + T_n + T_{m+n-1}.$$

That is L is the sum of triangular numbers, and triangular numbers describe the process of increasing the number of central places in **a triangular network**. More precisely – they inform how many central places form (fill) the studied equilateral triangle. On the other hand, the Lösch formula is their generalization to the entire hierarchical structure.

The final conclusion is that in the Theory of Central Places, the Lösch numbers can be understood in two ways:

1. algebraically (numbers L)
2. geometrically (T numbers).

For the hierarchy of Lösch and Christaller is rooted in the geometry of the triangular lattice.

Due to idealized assumptions and artificial geometrization of spatial market influences, CPT has been critically evaluated in many works, e.g. Saey (1973), Rushton (1971), White (1977), Parr (2017).

On the other hand, the development of the theory of gravitational spatial interactions related to the concept of shaping gravitational attractors based on the Newtonian equation of motion and deterministic chaos – confirmed the high degree of incompatibility of CPT with reality.

This difference is easily noticeable in Figure 2, which present the gravitational attractors of the central places. It can be concluded that the distribution of market, considering three different networks comprising central places, is in line with the CPT only in the case of extremely high space resistance – transport costs. However, as we know, in real economic activity, there is a constant tendency to reduce them, not to increase them.

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