

Viscoelastic wave scattering in near-surface soils: integral method for deformed layers within a stratified background

Introduction

Non-destructive imaging of the near-surface zone aims to characterize various physical properties of the soil by analyzing wave propagation through inversion processes such as Full Waveform Inversion (FWI). Since the forward model is called upon numerous times during these inversions, its computational speed remains the critical factor limiting the overall reconstruction rate (Virieux and Operto, 2009).

To address this need, very fast semi-analytical forward models based on the stratification hypothesis have been developed, applicable to wave propagation across different physical domains, including mechanics and electromagnetics (Ursin, 1983). In the context of mechanical wave propagation, we have implemented a code based on the Global Stiffness Matrix (GSM) method for 2D viscoelastic stratified media (Lowe, 1995). However, the stratification hypothesis reaches its limits in the presence of marked lateral heterogeneities (Sripanich et al., 2019; Patzold et al., 2008), justifying the use of extended methods that account for inclusions to overcome this constraint.

Classical approaches in related fields, such as acoustics and more recently elastic media (Shekhar et al., 2024), typically employ a homogeneous background and solve the resulting integral system either via the Born series or through more elaborated iterative schemes (Xiang et al., 2022). While the integral kernel in this case is analytical and fast to compute, the perturbation of medium parameters extends over the entire domain, requiring a dense discretization that makes matrix inversion computationally expensive. To mitigate this, the Born series approximation is often used; however, it may fail to converge for strong contrasts, necessitating specialized convergence enforcement techniques (Osnabrugge et al., 2016). Our strategy diverges from this paradigm by selecting a stratified background model computed via the GSM method. Instead of relying on a Born series with an homogeneous and infinite background, our approach treats the geometric deviation as a localized inclusion. This reduces the problem to solving a small, dense linear system restricted to the inclusion volume. Consequently, we obtain an exact solution for the perturbed field with minimal computational cost, completely bypassing the convergence issues inherent to iterative series expansions in complex media.

In this article, we take a first step in this direction by proposing a 2D model valid for a single density contrast, which also affects wave velocities, while the Lamé parameters remain constant. Although the Born approximation has already been employed in this specific context, for instance for cavity detection (Kaslihar, 2007; Filippi et al., 2019), our approach is designed as a foundational step toward handling contrasts in all three physical properties (ρ, λ, μ) currently under development. This method is applied to a model consisting of a layer overlying a half-space, where the interface forms a local bulge protruding into the semi-infinite medium.

Theory

The wave propagation is governed by the frequency-domain elastodynamic Navier equation:

$$(\tilde{\lambda} + \tilde{\mu})\nabla(\nabla \cdot \mathbf{u}) + \tilde{\mu}(\mathbf{r})\nabla^2 \mathbf{u} + \omega^2 \rho(\mathbf{r})\mathbf{u} = \mathbf{f}\delta_{\mathbf{r}}, \quad (1)$$

where $\mathbf{u}$ is the displacement field, ω the angular frequency and $\mathbf{f}\delta_{\mathbf{r}}$ is a punctual force. In this framework, each point $\mathbf{r} = x\hat{e}_x + z\hat{e}_z$ of the medium is characterized by its density $\rho(\mathbf{r})$ and complex Lamé parameters $\tilde{\lambda}$ and $\tilde{\mu}$, with x representing the horizontal distance and z the depth (negative downwards). The complexity of these parameters accounts for viscoelastic attenuation. This formulation remains valid for any spatial distribution of density, as the Lamé parameters are held constant in this study.

In our approach, the background medium corresponds to a stratified configuration for which we have developed an efficient semi-analytical solver (Knopoff, 1964; Mesgouez and Lefeuvre-Mesgouez, 2009). The objective is to deduce the wavefield in the actual medium, defined as this background perturbed by a local inclusion, without solving the full problem from scratch.

We introduce a fictitious unit point source located at any measurement point $\mathbf{r}_m$ with a force vector $\mathbf{f}_m$ oriented along either the horizontal (x) or vertical (z) direction. In the following notation, the subscript m refers to quantities associated with this fictitious source at $\mathbf{r}_m$, while the subscript l denotes the field associated with the actual source location $\mathbf{r}_l$. For instance, $\mathbf{u}_{a,l}(\mathbf{r}_m)$ denotes the predicted displacement field in the actual medium, generated by a source located at $\mathbf{r}_l$ and observed at position $\mathbf{r}_m$. This framework allows us to probe the medium response via reciprocity. By writing Eq. 1 for both the background medium (subscript b) and the actual medium (subscript a), and assuming a contrast in density ($\chi = \rho_a - \rho_b$) only, we derive the following integral equation:

$$\mathbf{u}_{a,l}(\mathbf{r}_m) = \mathbf{u}_{b,l}(\mathbf{r}_m) + \int_V \bar{\mathbf{G}}_b(\mathbf{r}_m, \mathbf{r}) \cdot [\chi(\mathbf{r}) \mathbf{u}_{a,l}(\mathbf{r})] dV, \quad (2)$$

In this equation, V denotes the inclusion volume, and $\bar{\mathbf{G}}_b(\mathbf{r}_m, \mathbf{r})$ is the dyadic Green's tensor. Its first row corresponds to the horizontal and vertical displacement components (u_x, u_z) at $\mathbf{r}$ generated by a horizontal unit force at $\mathbf{r}_m$, while its second row represents the field components resulting from a vertical unit force at the same locations. This 2×2 operator thus fully captures the PSV wavefield response to both polarization states.

To solve this numerically, we discretize the inclusion volume V into N_I cells and consider N_M observation points. We introduce the following vectors and matrices:

- $\mathbf{u}_{a,l}$ and $\mathbf{u}_{b,l}$ ($2N_M \times 1$): Displacement vectors at the receiver locations for the actual and background media, respectively.
- $\mathbf{u}_a$ and $\mathbf{u}_b$ ($2N_I \times 1$): Total and background fields evaluated *inside* the inclusion cells.
- $\bar{\mathbf{G}}$ ($2N_I \times 2N_I$): The interaction matrix where each 2×2 block $\bar{\mathbf{G}}_{ij}$ corresponds to the Green's tensor between cell j and cell i within the inclusion.
- $\bar{\mathbf{G}}_{M \rightarrow I}$ ($2N_I \times 2N_M$): The propagation matrix where each 2×2 block connects a source location j on the measurement line to a receiver cell i within the inclusion.
- ξ ($2N_I \times 2N_I$): A diagonal matrix containing the contrast values $\chi(\mathbf{r}_k)$ scaled by the cell volume ΔV_k .

The continuous integral equation thus transforms into a coupled linear system:

$$\mathbf{u}_a = \mathbf{u}_b + \bar{\mathbf{G}} \xi \mathbf{u}_a \quad \text{state equation} \quad (3a)$$

$$\mathbf{u}_{a,l} = \mathbf{u}_{b,l} + \bar{\mathbf{G}}_{M \rightarrow I} \xi \mathbf{u}_a \quad \text{observation equation} \quad (3b)$$

Equation (3a) allows us to solve for the total field inside the inclusion $\mathbf{u}_a = (\mathbf{I} - \bar{\mathbf{G}} \xi)^{-1} \mathbf{u}_b$, which is then substituted into Eq. (3b) to retrieve the scattered field at the receivers. Visually, the system in Eq. (3a) expands as:

$$\begin{pmatrix} \mathbf{u}_a^{(1)} \\ \vdots \\ \mathbf{u}_a^{(N_I)} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_b^{(1)} \\ \vdots \\ \mathbf{u}_b^{(N_I)} \end{pmatrix} + \begin{pmatrix} \bar{\mathbf{G}}_{11} & \cdots & \bar{\mathbf{G}}_{1N_I} \\ \vdots & \ddots & \vdots \\ \bar{\mathbf{G}}_{N_I 1} & \cdots & \bar{\mathbf{G}}_{N_I N_I} \end{pmatrix} \begin{pmatrix} \chi^{(1)} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \chi^{(N_I)} \end{pmatrix} \begin{pmatrix} \mathbf{u}_a^{(1)} \\ \vdots \\ \mathbf{u}_a^{(N_I)} \end{pmatrix}, \quad (4)$$

where each term $\mathbf{u}^{(k)}$ is a 2-component vector $(u_x, u_z)^T$ at cell k .

Results

To validate the robustness and define the limits of the proposed code, preliminary tests were conducted on complex geometries, including two large 10-meter disks, with dimensions comparable to the wavelength, embedded in a semi-infinite medium. While the semi-analytical approach is capable of modeling arbitrary interface shapes, a semi-circular geometry was selected for this benchmark to facilitate a direct and simplified comparison with Finite Element Method (FEM) references. The study case, illustrated in Figure 1, applies the method to a model consisting of a layer overlying a half-space, where the interface

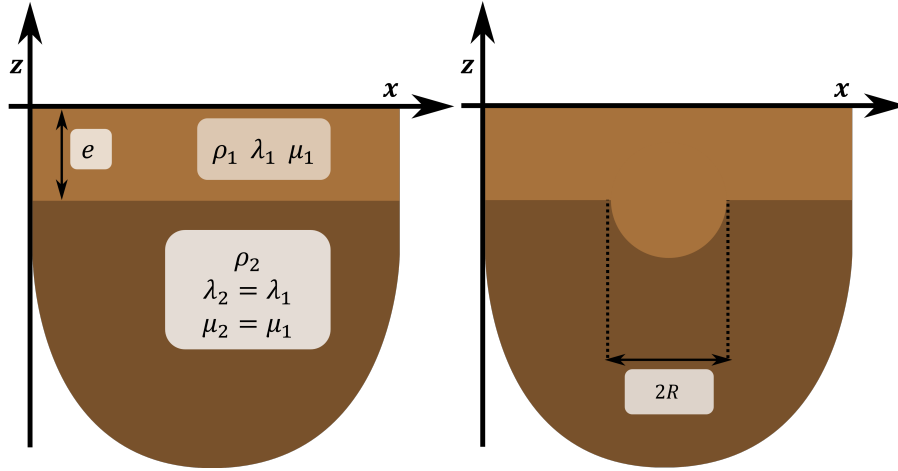


Figure 1 Schematic of the model. Left: Reference stratified background (flat layer of thickness e). Right: Perturbed model with a semi-circular interface deformation of radius R , treated as an inclusion.

forms a local semi-circular bulge of radius $R = 1$ m protruding into the substrate. The **background reference** is a perfectly stratified medium (flat layer of thickness e on a half-space), and the perturbation corresponds solely to the local deformation of this interface. The substrate (Medium 1) features fixed Lamé parameters (λ_1, μ_1) derived from viscoelastic wave speeds, while Medium 2 maintains these same stiffness parameters but varies in density to alter its wave propagation characteristics; specifically, Medium 1 is assigned a density $\rho_1 = 1500$ kg/m³ with P- and S-wave velocities of 1000 m/s and 500 m/s (at 100 Hz) with 5% loss to define the reference stiffness, while Medium 2 retains these parameters with a higher density $\rho_2 = 2500$ kg/m³, inducing velocity reduction.

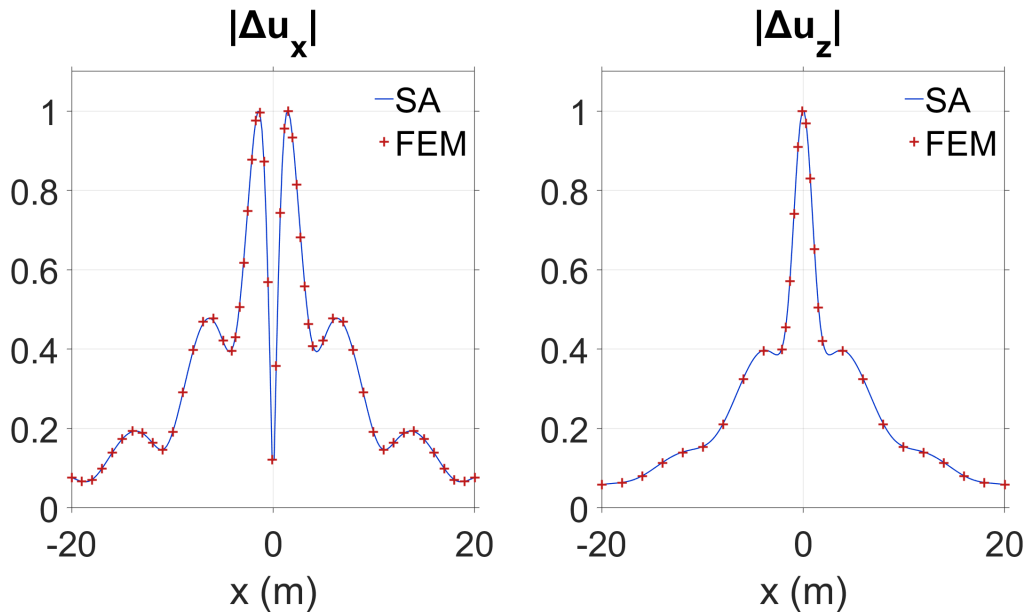


Figure 2 Comparison of Semi-Analytical (SA) and Finite Element Method (FEM) results. The left panel displays the horizontal perturbation field (Δu_x), while the right panel shows the vertical perturbation field (Δu_z). All values are normalized and presented in arbitrary units

Figure 2 displays the perturbation field, defined as the difference between the total field in the model with the deformed interface and the background field of the perfectly stratified medium. The computational efficiency of the semi-analytical method is striking: obtaining the converged solution for both FEM models (the reference stratified case and the deformed case) required approximately two minutes using 24 CPU cores. In contrast, the semi-analytical code completed the same calculation in less than one

second. Convergence was achieved with only $N_I = 5$ points within the inclusion and 200 points along the surface. To illustrate the marginal gain in precision versus cost, increasing the internal discretization to 30 points raised the computation time to merely 6 seconds, confirming the method's suitability for rapid parametric studies.

Conclusion

This study demonstrates that modeling interface deformations as localized inclusions within a stratified background offers a drastic computational advantage over full-domain numerical discretization. By restricting the mesh to the perturbation only, the proposed semi-analytical method solves the problem within one second, whereas a complete Finite Element Method (FEM) simulation requires approximately two minutes for the same accuracy. This efficiency, achieved without the convergence constraints of Born series in layered media, establishes a robust foundation for future extensions to complex contrasts and rapid full-waveform inversion.

Acknowledgements

The experimental component of this study was supported by the project AquaGeo ϕ (2023–2026), funded by the Provence-Alpes-Côte d'Azur Region (France), Avignon University, and the ANR Carnot Eau & Environnement program.

References

- Filippi, C., Leparoux, D., Grandjean, G., Bitri, A. and Côte, Ph. [2019] New robust observables on Rayleigh waves affected by an underground cavity: from numerical to experimental modelling. *Geophysical Journal International*, 218(3), 1903–1918.
- Kasilar, A. [2007] Inverse scattering of surface waves: imaging of near-surface heterogeneities. *Geophysical Journal International*, 171(1), 352–367.
- Knopoff, L. [1964] A matrix method for elastic wave problems. *Bulletin of the Seismological Society of America*, 54(1), 431–438.
- Lowe, M.J.S. [1995] Matrix techniques for modeling ultrasonic waves in multilayered media. *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control*, 42(4), 525–542.
- Mesgouez, A. and Lefeuvre-Mesgouez, G. [2009] Transient solution for multilayered poroviscoelastic media obtained by an exact stiffness matrix formulation. *International Journal for Numerical and Analytical Methods in Geomechanics*, 33(17), 1911–1931.
- Osnabrugge, G., Leedumrongwatthanakun, S. and Vellekoop, I.M. [2016] A convergent Born series for solving the inhomogeneous Helmholtz equation in arbitrarily large media. *Journal of Computational Physics*, 322, 113–124.
- Patzold, S., Mertens, F.M., Bornemann, L., Koleczek, B., Franke, J., Feilhauer, H. and Welp, G. [2008] Soil heterogeneity at the field scale: a challenge for precision crop protection. *Precision Agriculture*, 9(6), 367–390.
- Shekhar, U., Jakobsen, M., Iversen, E., Berre, I. and Radu, F.A. [2024] Microseismic wavefield modelling in anisotropic elastic media using integral equation method. *Geophysical Prospecting*, 72(2), 403–423.
- Sripanich, Y., Fomel, S. and Stovas, A. [2019] Effects of lateral heterogeneity on time-domain processing parameters. *Geophysical Journal International*, 219(2), 1181–1201.
- Ursin, B. [1983] Review of elastic and electromagnetic wave propagation in horizontally layered media. *Geophysics*, 48(8), 1063–1081.
- Virieux, J. and Operto, S. [2009] An overview of full-waveform inversion in exploration geophysics. *Geophysics*, 74, WCC1–WCC26.
- Xiang, K., Eikrem, K.S., Jakobsen, M. and Nævdal, G. [2022] Homotopy scattering series for seismic forward modelling with variable density and velocity. *Geophysical Prospecting*, 70(1), 3–18.