Is multiphase energy-conserving scheme really energy-conserving and accurate?

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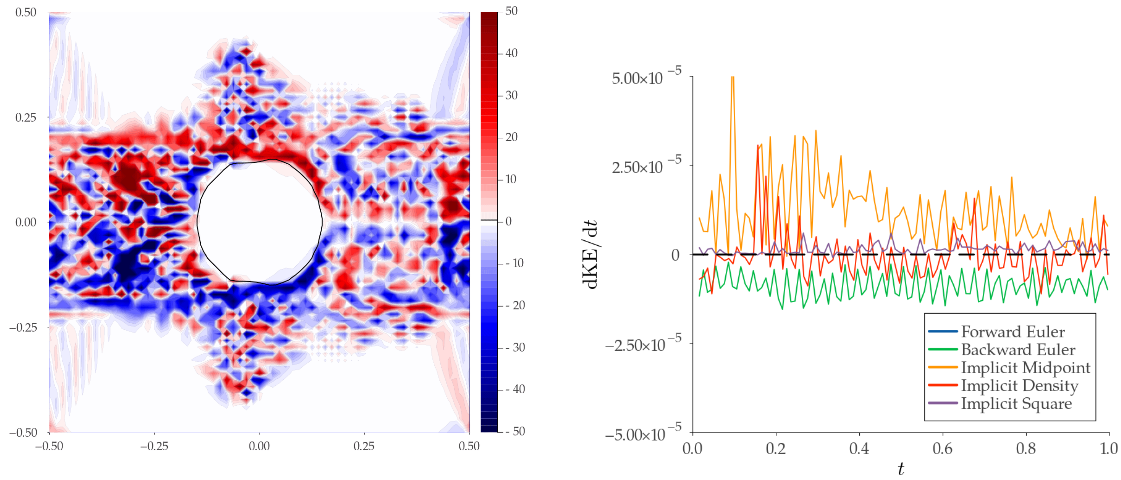
Traditional velocity-based schemes for incompressible multiphase flows are often unstable, particularly for high-density-contrast flows.[1] A class of momentum-preserving Consistent Mass-Momentum Transport Methods (CMOM)[2,3] addresses this problem by adopting a momentum-based formulation and ensuring consistency between mass and momentum fluxes. Among these, we focus on the kinetic energy-conserving subclass. Previous work[4,5] demonstrated that skew-symmetric momentum advection operators conserve kinetic energy in semi-discrete systems. However, their derivations overlook temporal discretization and rely on high-order or symplectic schemes designed for single-phase flows[6]. This study mathematically proves that traditional symplectic temporal integrators cannot guarantee kinetic energy conservation in fully discrete systems and proposes a modified implicit midpoint scheme validated using the traveling droplet test case, characterized by high-density ratios and strong velocity gradients (see Figure 1).

Exact energy conservation, while theoretically desirable, imposes overly strict constraints on the flow field, leading to nonphysical oscillations when advecting non-smooth fields without numerical diffusion. To balance conservation and accuracy, we introduce an energy-bounded minimal-dissipation scheme. This approach relaxes time integration to an explicit formulation and uses carefully selected local diffusion to remove excess energy from explicit time integration and nonphysical flow features. These improvements enable stable, efficient, and accurate simulations of high-density-contrast multiphase flows.



(a) Vorticity field after one convection cycle of implicit density-rooted time integration.

(b) Rate of change of kinetic energy.

Figure 1: Flow field and energy rate of change for the traveling droplet test case ($\rho_{\text{air}}/\rho_{\text{water}} = 10^{-4}$) under different variants of the implicit midpoint method. The vorticity field highlights velocity oscillations caused by the advection of non-smooth fields without any artificial diffusion such as flux limiter. In the energy evolution graph, the purple line represents the implicit method expected to conserve energy exactly, with fluctuations likely arising from implicit iterative solver tolerance and `Float32` precision limitations.

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