

A numerical study of the generalized surface quasigeostrophic equation

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We present numerical results on the generalized surface quasigeostrophic equation (gSQG)

$$\partial_t \theta + \gamma \Lambda^\alpha \theta + u \cdot \nabla \theta = g,$$

$$u = \nabla^\perp \psi, \quad \Delta \psi = \Lambda^\beta \theta, \quad \text{per. BCs on } [0, L]^2, \quad L = 2\pi$$

where $0 < \alpha \leq 2$, $0 \leq \beta < 2$, and Λ denotes the fractional Laplacian operator $(-\Delta)^{\frac{1}{2}}$. We test the extent to which several features of turbulence for the 2D Navier–Stokes equations hold for the gSQG, after a suitable rescaling. This includes analogues of Kraichnan’s power law the energy spectrum, a direct enstrophy cascade, and dissipation law. We also test the efficacy of data assimilation by nudging for this family of equations. An empirically determined resolution for the data is compared to a rigorous sufficient condition¹ in the particular case of the subcritical SQG ($\alpha \in (1, 2)$, $\beta = 1$).

The sample plot in Figure 1 shows the quotient of the enstrophy dissipation rate $\eta = \frac{\gamma}{L^2} \langle |\Lambda^{\frac{\alpha}{2}} \theta|^2 \rangle$ and the enstrophy flux $\langle \mathfrak{E}_\kappa(\theta) \rangle = \frac{\gamma}{L^2} \langle |\Lambda^{\frac{\alpha}{2}} Q_\kappa \theta|^2 \rangle$, where $Q_\kappa \theta = \sum_{|k| > \kappa} \hat{\theta}_k e^{ik \cdot x}$, for seven values of (α, β) . This indicates how pronounced the enstrophy cascade is as well as the significance of the bound

$$1 - \left(\frac{\kappa}{\kappa_\sigma} \right)^\alpha \leq \frac{\langle \mathfrak{E}_\kappa \rangle}{\eta} \leq 1,$$

rescaled from one for the Navier–Stokes case² ($\alpha = 2, \beta = 0$).

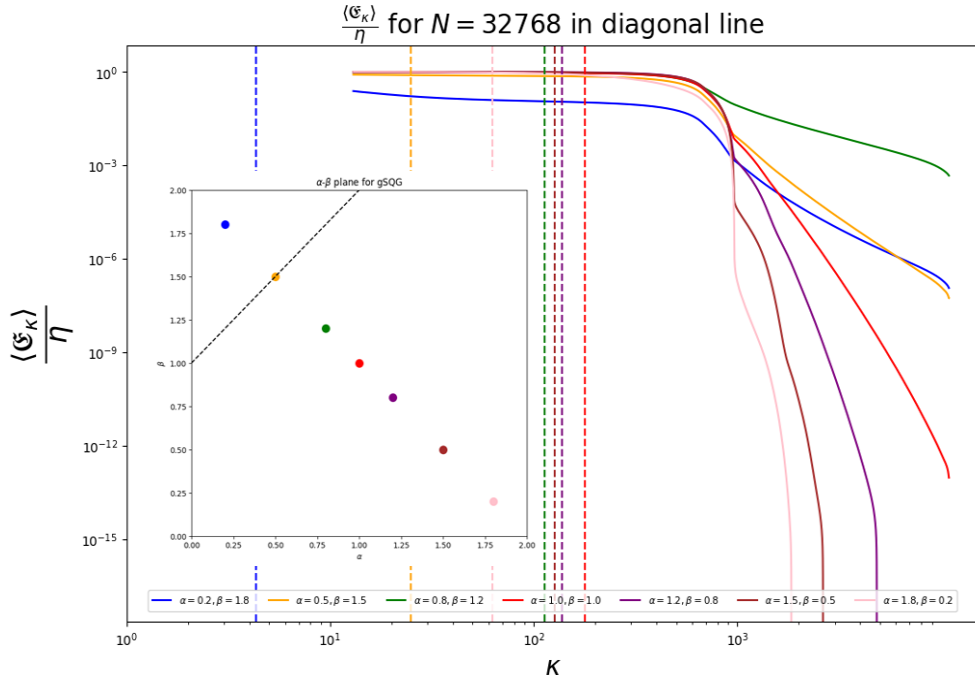


Figure 1: From a pseudospectral code using N^2 modes. The vertical lines indicate $\kappa_\sigma = \left(\frac{\langle |\Lambda^{\frac{\alpha}{2}} \theta|^2 \rangle}{\langle |\theta|^2 \rangle} \right)^{(1/\alpha)}$.

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¹ Jolly et al. *Adv. Nonlinear Stud.* **17**:167-192 (2017)

² Foias et al. *J. Stat. Phys.* **108**:591-645 (2002)