A Perturbative Correction to the Quasi-Linear Approximation for Stratified Turbulence

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Although ubiquitous in geophysical flows, strongly stratified turbulence gives rise to a highly anisotropic separation of scales that makes it infeasible to simulate directly. Chini et al.[1] demonstrated that a quasi-linear (QL) reduction of the Boussinesq equations is formally justified in the limit of strong stable density stratification. Quantitatively, this occurs when the (horizontal) Froude number $Fr := U/[NL] \to 0$; here, U and L are characteristic horizontal velocity and length scales, and N is a characteristic value of the buoyancy frequency. In this limit, there is a separation between the time scale of the fastest-growing stratified shear instabilities and the advection/vertical-shear time scale of the large-scale, pancake-like horizontal flow, causing the QL instabilities to self-tune to a state of near-marginal stability. These features enabled the authors to develop a hybrid eigenvalue/initial-value algorithm for efficiently simulating the resulting slow-fast QL system. For small but finite Fr (or sufficiently large Reynolds number $Re \doteq UL/\nu$, where ν is the kinematic viscosity), however, a wide range of fast modes may be excited nonlinearly, leading to a spectrally-local downscale energy cascade at odds with the assumptions of their model.

Here, we show that these additional modes can be perturbatively incorporated via a weakly nonlinear analysis about the *emergent* marginal stability manifold. The modified algorithm requires only $O(N_z^3)$ floating-point operations per horizontal grid point, where N_z is the vertical mesh resolution; to leading order in N_z, this is independent of the number of fluctuating modes, and only a small multiple of the cost of the single-mode algorithm. Notably, our work also reduces the computational cost of the single-mode algorithm by a factor of four, and our algorithm can be integrated adaptively with the single-mode algorithm or with other sub-grid-scale models. We demonstrate the efficacy of the extende 'nonlinear cascade' NCQL algorithm for 2D stratified Kolmogorov flow via direct comparisons with direct numerical simulation (DNS) of the problem, for parameters as extreme as $Fr = 0.06$, $Re_b = 60$, with $Re_b \doteq ReFr^2$ the buoyancy Reynolds number: the figure shows just such a comparison between the new algorithm and DNS, where the previously reported algorithm of Chini et al. (2022) diverges. The systematic extension of QL theory developed herein should prove advantageous for other shear flows for which the QL reduction is a useful model for inherently multiscale dynamics.

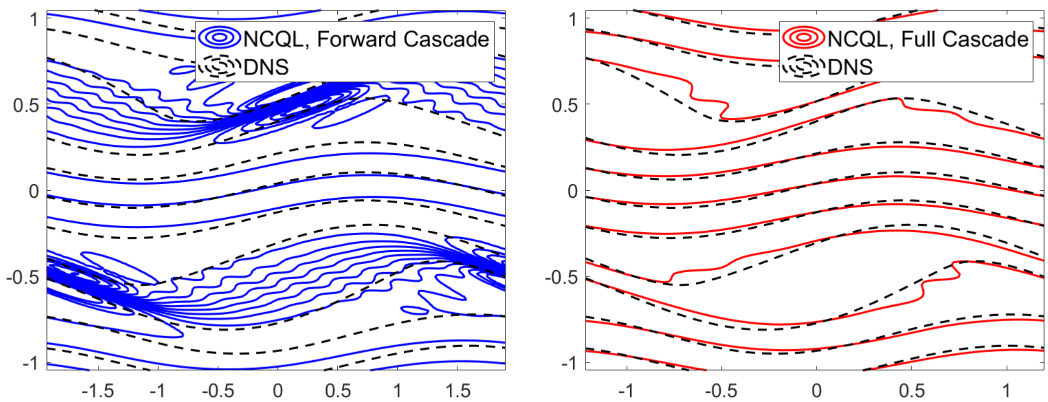

Figure 1: An example buoyancy field constructed by the (full) 'nonlinear cascade quasi-linear' NCQL algorithm, with $Re_b = 10$, $Fr = 0.02$, and $n = 4$, compared to that of the baseline 'forward' NCQL algorithm and a DNS of the same wavelength as the former (sampled at $t = 20.0$). The full cascade NCQL algorithm, which incorporates an upstream cascade among the fluctuating modes, settles to a steady-state that closely matches the DNS result, while the 'forward' NCQL algorithm diverges.

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[1] Chini et al., *J. Fluid Mech.* **933**, A22 (2022)