A rational constitutive law for the viscous stress tensor in the one-fluid model of two-phase flows

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The one-fluid model is routinely used to simulate two-phase flows with moving interfaces on a fixed grid, using approaches such as the Volume of Fluid (VOF) or the Level Set method. In this model, the momentum equation is assumed to hold throughout the flow domain, with the fluid properties varying in space according to the local composition of the two-phase medium. For instance, in the VOF approach, mass conservation implies that the local density of a two-phase medium involving two fluids with respective densities ρ_1 and ρ_2 must be defined as $\rho(C) = C\rho_1 + (1-C)\rho_2$, with C being the volume fraction of fluid 1 in the elementary control volume centered at the considered position. No such argument exists to guide the definition of the viscosity of the medium. Instead, *ad hoc* averaging rules are used, such as the arithmetic law $\lambda(C) = C\mu_1 + (1-C)\mu_2$, or the harmonic law $\kappa(C) = \mu_1\mu_2/((1-C)\mu_1 + C\mu_2)$, with μ_1 and μ_2 being the shear viscosities of the two fluids. We revisited this issue by considering the two-phase medium as an anisotropic fluid exhibiting a preferential direction along the local normal to the interface. We employed the theory of representations to obtain the generic form of the constitutive law, and determined its unknown coefficients through the matching conditions at the interface of a specific 2D flow in which the local velocity of each fluid results from the superposition of a constant shear and a constant elongation along the interface. Assuming a linear relationship between the viscous stress tensor, $\tilde{\Sigma}_{ij}$, and the strain-rate tensor, S_{ij}, then yields the general constitutive law $\tilde{\Sigma}_{ij} = 2\lambda(C)S_{ij} + 2(\kappa(C) - \lambda(C))[(S_{ik}n_k)n_j + n_i(S_{jk}n_k) - 2(n_k S_{kl} n_l)n_i n_j]$.
To compare predictions based on this constitutive law with those based on the above *ad hoc* expressions, we designed a specific facility aimed at producing a gravity-driven exchange flow within a long cylindrical tube closed at both ends. For this purpose, the lower and upper halves of the tube are initially filled with a heavy and a light fluid, respectively, with a small relative density difference $\Delta\overline{\rho} \equiv (\rho_1 - \rho_2)/(\rho_1 + \rho_2) \ll 1$, a large viscosity contrast $\mu_1/\mu_2 \gg 1$, and negligible interfacial tension. The tube is then turned upside down and the exchange flow sets in. The light fluid rises in the form of a central, nearly axisymmetric, finger surrounded by a thin descending film of heavy fluid. Conversely, the heavy fluid goes down in the form of a highly asymmetric sinuous filament with a bulbous end. We recorded the position of the rising and descending tips with a high-speed camera. In parallel, we simulated the experimental configuration, matching as closely as possible the measured fluid properties so as to compare the predicted tip positions with those determined experimentally. This comparison is illustrated in figure 1. The *ad hoc* arithmetic law $\tilde{\Sigma}_{ij} = 2\lambda(C)S_{ij}$ is seen to severely under-predict the speed of the descending tip, while the harmonic law $\tilde{\Sigma}_{ij} = 2\kappa(C)S_{ij}$ tends to over-predict it. In contrast, the rational constitutive law properly captures the evolution of both tips. This comparison demonstrates that accurate predictions of flows in which viscous stresses control the interface dynamics require the anisotropic nature of the two-phase medium to be explicitly taken into account in the constitutive law for the viscous stress tensor.

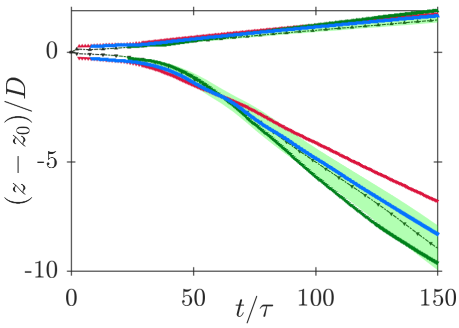

Figure 1: Positions of the ascending and descending tips vs time in an exchange flow with $\mu_1/\mu_2 \approx 33$ and $\rho_1(2\Delta\overline{\rho}gD^3)^{1/2}/\mu_1 \approx 0.41$, with D the tube diameter and g denoting gravity. Times are normalized by the viscous scale $\tau = \mu_1/((\rho_1 - \rho_2)gD)$. The green region corresponds to the enveloppe of the experimental data, a typical example of which is shown by the dash-triangle line. The blue line is the prediction obtained with the rational constitutive law, while the red and green lines are those obtained with the *ad hoc* arithmetic and harmonic viscosity models, respectively.

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